Correct-by-Construction Cryptographic Arithmetic in Coq

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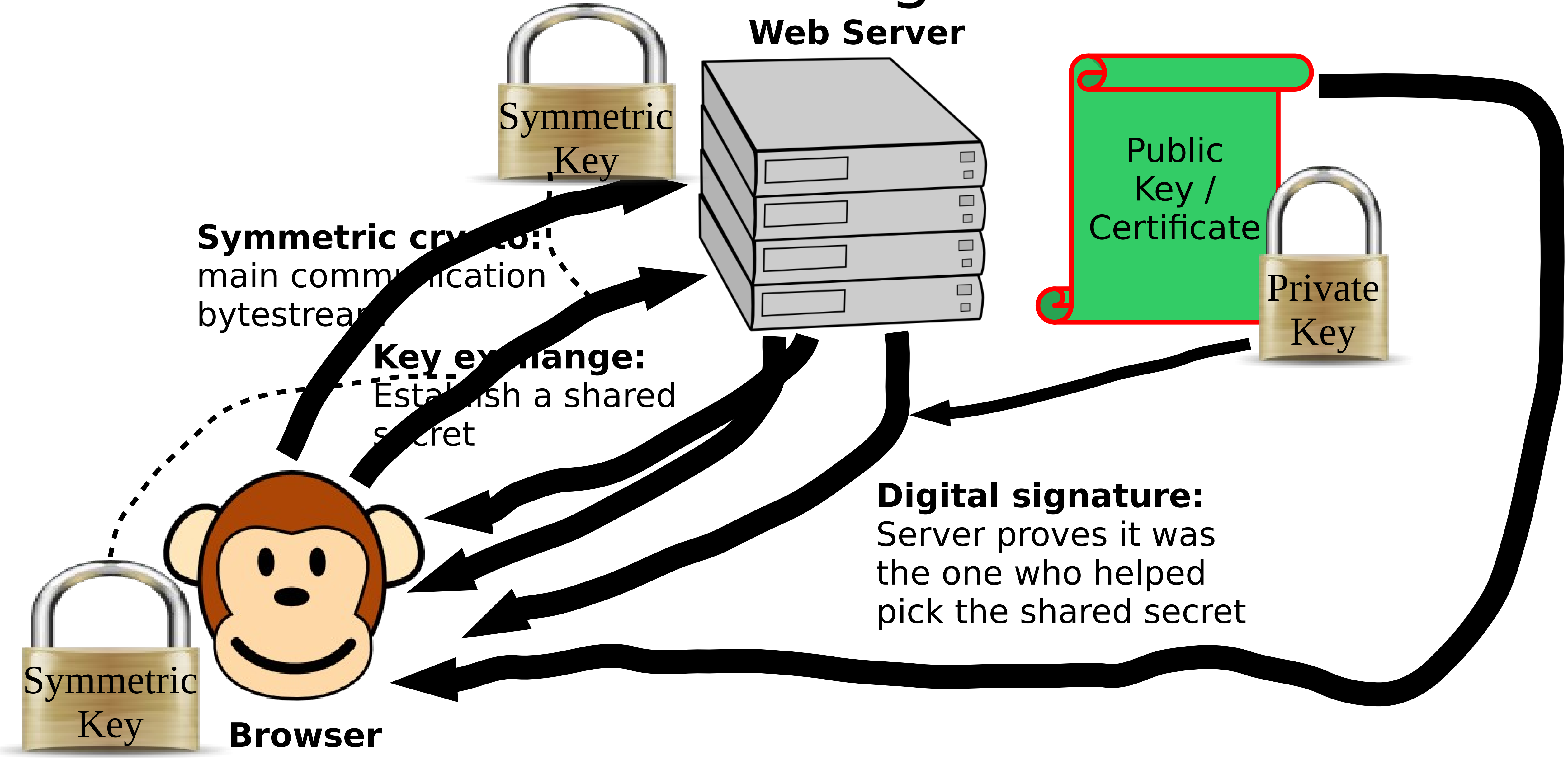
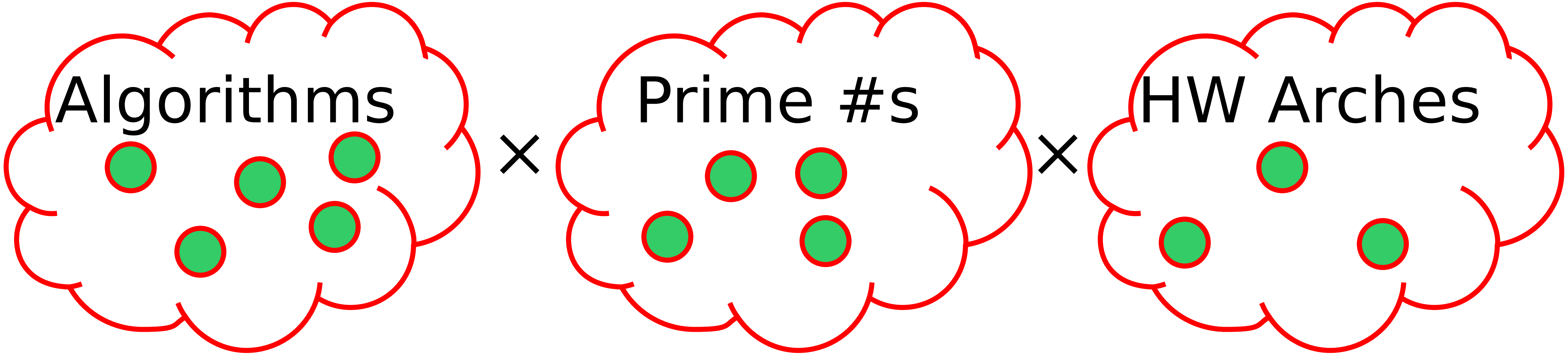
Web Browsing with SSL



About the First Two Stages (Public-Key Crypto)

- Public-key stages only run once per session, but, with many small HTTPS connections common in practice, their performance is still important.
- Balancing correctness and performance is also more challenging for the public-key algorithms.
  - Primarily: **big-integer modular arithmetic**

But the experts know how to do all this, right?



Labor-intensive adaptation, with each combination taking significant expert effort.

We introduced *Fiat Cryptography*.

- An automatic generator for this kind of code,
- with correctness proofs in the Coq theorem prover.
- Adopted for small but important parts of TLS implementations in both Chrome and Firefox, plus a number of blockchain systems, etc.

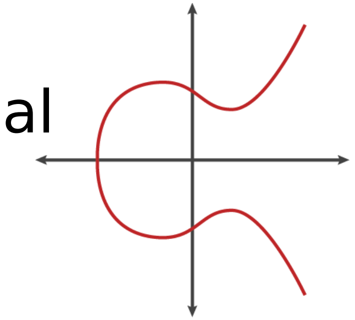
Correct-by-Construction Cryptography

Abstract security property

“Knowledge of the secret key is needed to produce a signature in polynomial time.”

protocol verification

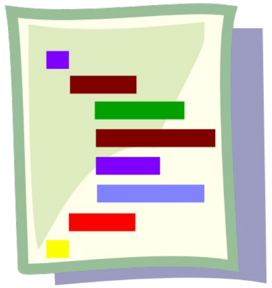
Mathematical algorithm



$$y^2 = x^3 - x + 1$$

implementation synthesis

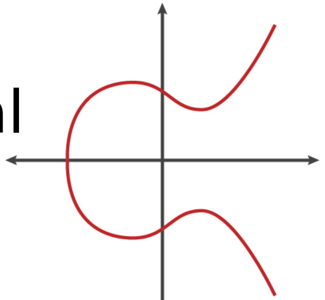
Low-level code



specialized assembly code

Correct-by-Construction Cryptography

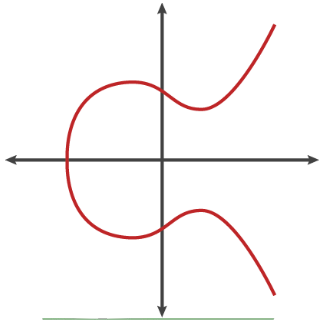
Mathematical algorithm



point = (x, y)

classic verification of functional programs

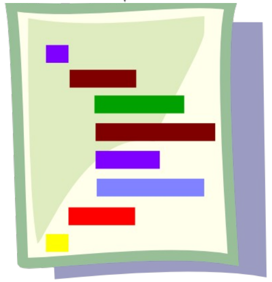
Optimized point format



point = (x, y, z, t)

classic verification of functional programs

High-level modular arithmetic

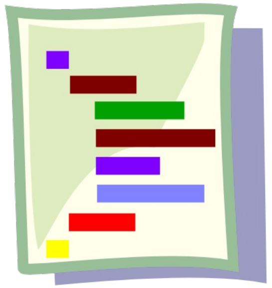


$x = x_0, x_1, \ldots, x_n$

(mathematical integers)

compile-time code specialization

Low-level code



specialized low-level code

(assumes fixed set of integer sizes)

compiler verification

Generated Code

Squaring a number (64-bit)

```
λ '(x7, x8, x6, x4, x2)%core,
uint64_t x9 = x2 * 0x2;
uint64_t x10 = x4 * 0x2;
uint64_t x11 = x6 * 0x2 * 0x13;
uint64_t x12 = x7 * 0x13;
uint64_t x13 = x12 * 0x2;
uint128_t x14 = (uint128_t) x2 * x2 + (uint128_t) x13 * x4 + (uint128_t) x11 * x8;
uint128_t x15 = (uint128_t) x9 * x4 + (uint128_t) x13 * x6 + (uint128_t) x8 * (x8 * 0x13);
uint128_t x16 = (uint128_t) x9 * x6 + (uint128_t) x4 * x4 + (uint128_t) x13 * x8;
uint128_t x17 = (uint128_t) x9 * x8 + (uint128_t) x10 * x6 + (uint128_t) x7 * x12;
uint128_t x18 = (uint128_t) x9 * x7 + (uint128_t) x10 * x8 + (uint128_t) x6 * x6;
uint64_t x19 = (uint64_t) (x14 >> 0x33);
uint64_t x20 = (uint64_t) x14 & 0x7ffffffffffff;
uint128_t x21 = x19 + x15;
uint64_t x22 = (uint64_t) (x21 >> 0x33);
uint64_t x23 = (uint64_t) x21 & 0x7ffffffffffff;
uint128_t x24 = x22 + x16;
uint64_t x25 = (uint64_t) (x24 >> 0x33);
uint64_t x26 = (uint64_t) x24 & 0x7ffffffffffff;
uint128_t x27 = x25 + x17;
uint64_t x28 = (uint64_t) (x27 >> 0x33);
uint64_t x29 = (uint64_t) x27 & 0x7ffffffffffff;
uint128_t x30 = x28 + x18;
uint64_t x31 = (uint64_t) (x30 >> 0x33);
uint64_t x32 = (uint64_t) x30 & 0x7ffffffffffff;
uint64_t x33 = x20 + 0x13 * x31;
uint64_t x34 = x33 >> 0x33;
uint64_t x35 = x33 & 0x7ffffffffffff;
uint64_t x36 = x34 + x23;
uint64_t x37 = x36 >> 0x33;
uint64_t x38 = x36 & 0x7ffffffffffff;
return (Return x32, Return x29, x37 + x26, Return x38, Return x35))
```

Squaring a number (32-bit)

```
λ '(x17, x18, x16, x14, x12, x10, x8, x6, x4, x2)%core,
uint64_t x19 = (uint64_t) x2 * x2;
uint64_t x20 = (uint64_t) (0x2 * x2) * x4;
uint64_t x21 = 0x2 * ((uint64_t) x4 * x4 + (uint64_t) x2 * x6);
uint64_t x22 = 0x2 * ((uint64_t) x4 * x6 + (uint64_t) x2 * x8);
uint64_t x23 = (uint64_t) x6 * x6 + (uint64_t) (0x4 * x4) * x8 + (uint64_t) (0x2 * x2) * x10;
uint64_t x24 = 0x2 * ((uint64_t) x6 * x8 + (uint64_t) x4 * x10 + (uint64_t) x2 * x12);
uint64_t x25 = 0x2 * ((uint64_t) x8 * x8 + (uint64_t) x6 * x10 + (uint64_t) x2 * x14 + (uint64_t) (0x2 * x4) * x12);
uint64_t x26 = 0x2 * ((uint64_t) x8 * x10 + (uint64_t) x6 * x12 + (uint64_t) x4 * x14 + (uint64_t) x2 * x16);
uint64_t x27 = (uint64_t) x10 * x10 + 0x2 * ((uint64_t) x6 * x14 + (uint64_t) x2 * x18 + 0x2 * ((uint64_t) x4 * x16 + (uint64_t) x8 * x12));
uint64_t x28 = 0x2 * ((uint64_t) x10 * x12 + (uint64_t) x8 * x14 + (uint64_t) x6 * x16 + (uint64_t) x4 * x18 + (uint64_t) x2 * x17);
uint64_t x29 = 0x2 * ((uint64_t) x12 * x12 + (uint64_t) x10 * x14 + (uint64_t) x6 * x18 + 0x2 * ((uint64_t) x8 * x16 + (uint64_t) x4 * x17));
uint64_t x30 = 0x2 * ((uint64_t) x12 * x14 + (uint64_t) x10 * x16 + (uint64_t) x8 * x18 + (uint64_t) x6 * x17);
uint64_t x31 = (uint64_t) x14 * x14 + 0x2 * ((uint64_t) x10 * x18 + 0x2 * ((uint64_t) x12 * x16 + (uint64_t) x8 * x17));
uint64_t x32 = 0x2 * ((uint64_t) x14 * x16 + (uint64_t) x12 * x18 + (uint64_t) x10 * x17);
uint64_t x33 = 0x2 * ((uint64_t) x16 * x16 + (uint64_t) x14 * x18 + (uint64_t) (0x2 * x12) * x17);
uint64_t x34 = 0x2 * ((uint64_t) x16 * x18 + (uint64_t) x14 * x17);
uint64_t x35 = (uint64_t) x18 * x18 + (uint64_t) (0x4 * x16) * x17;
uint64_t x36 = (uint64_t) (0x2 * x18) * x17;
uint64_t x37 = (uint64_t) (0x2 * x17) * x17;
uint64_t x38 = x27 + x37 << 0x4;
uint64_t x39 = x38 + x37 << 0x1;
uint64_t x40 = x39 + x37;
uint64_t x41 = x26 + x36 << 0x4;
uint64_t x42 = x41 + x36 << 0x1;
uint64_t x43 = x42 + x36;
uint64_t x44 = x25 + x35 << 0x4;
uint64_t x45 = x44 + x35 << 0x1;
uint64_t x46 = x45 + x35;
uint64_t x47 = x24 + x34 << 0x4;
uint64_t x48 = x47 + x34 << 0x1;
uint64_t x49 = x48 + x34;
uint64_t x50 = x23 + x33 << 0x4;
uint64_t x51 = x50 + x33 << 0x1;
uint64_t x52 = x51 + x33;
uint64_t x53 = x22 + x32 << 0x4;
uint64_t x54 = x53 + x32 << 0x1;
uint64_t x55 = x54 + x32;
uint64_t x56 = x21 + x31 << 0x4;
uint64_t x57 = x56 + x31 << 0x1;
uint64_t x58 = x57 + x31;
uint64_t x59 = x20 + x30 << 0x4;
uint64_t x60 = x59 + x30 << 0x1;
uint64_t x61 = x60 + x30;
uint64_t x62 = x19 + x29 << 0x4;
uint64_t x63 = x62 + x29 << 0x1;
uint64_t x64 = x63 + x29;
uint64_t x65 = x64 >> 0x1a;
uint32_t x66 = (uint32_t) x64 & 0x3ffffff;
uint64_t x67 = x65 + x61;
uint64_t x68 = x67 >> 0x19;
uint32_t x69 = (uint32_t) x67 & 0x1ffffff;
uint64_t x70 = x68 + x58;
uint64_t x71 = x70 >> 0x1a;
uint32_t x72 = (uint32_t) x70 & 0x3ffffff;
uint64_t x73 = x71 + x55;
uint64_t x74 = x73 >> 0x19;
uint32_t x75 = (uint32_t) x73 & 0x1ffffff;
uint64_t x76 = x74 + x52;
uint64_t x77 = x76 >> 0x1a;
uint32_t x78 = (uint32_t) x76 & 0x3ffffff;
uint64_t x79 = x77 + x49;
uint64_t x80 = x79 >> 0x19;
uint32_t x81 = (uint32_t) x79 & 0x1ffffff;
uint64_t x82 = x80 + x46;
uint32_t x83 = (uint32_t) (x82 >> 0x1a);
uint32_t x84 = (uint32_t) x82 & 0x3ffffff;
uint64_t x85 = x83 + x43;
uint32_t x86 = (uint32_t) (x85 >> 0x19);
uint32_t x87 = (uint32_t) x85 & 0x1ffffff;
uint64_t x88 = x86 + x40;
uint32_t x89 = (uint32_t) (x88 >> 0x1a);
uint32_t x90 = (uint32_t) x88 & 0x3ffffff;
uint64_t x91 = x89 + x28;
uint32_t x92 = (uint32_t) (x91 >> 0x19);
uint32_t x93 = (uint32_t) x91 & 0x1ffffff;
uint64_t x94 = x66 + (uint64_t) 0x13 * x92;
uint32_t x95 = (uint32_t) (x94 >> 0x1a);
uint32_t x96 = (uint32_t) x94 & 0x3ffffff;
uint32_t x97 = x95 + x69;
uint32_t x98 = x97 >> 0x19;
uint32_t x99 = x97 & 0x1ffffff;
return (Return x93, Return x90, Return x87, Return x84, Return x81, Return x78, Return x75, x98 + x72, Return x99, Return x96))
```

Surprising (?) Fact About Modular Arithmetic

Different prime moduli have dramatically different efficiency with best code on commodity processors.

$2^{255} - 19$ is a popular choice for relatively easy implementation.
General pattern: $2^k - c$, for $c << 2^k$. (Called *pseudo-Mersenne*.)
Example of a fast operation: *modular reduction*

$t = x + 2^k y \pmod{2^k - c}$ too big to fit below the modulus!
$= x + (2^k - c + c)y \pmod{2^k - c}$
$= x + \cancel{(2^k - c)y} + cy \pmod{2^k - c}$
$= x + cy \pmod{2^k - c}$

Representing Numbers mod $2^{255} - 19$

t result of multiplying two numbers in the prime field, so **510 bits wide**

$= t_0\ t_1\ t_2\ t_3\ t_4\ t_5\ t_6\ t_7$ each “digit” fits in 64-bit register

$= (t_0 + 2^{64}\ t_1 + \ldots) + 2^{256}\ (t_4 + 2^{64}\ t_5 + \ldots)$

darn, that's 2^{256}, not 2^{255}, so we can't use that reduction trick!

However.... $51 \times 10 = 510$.

$t = (t_0 + 2^{51}\ t_1 + \ldots) + 2^{255}\ (t_5 + 2^{51}\ t_6 + \ldots)$

champion rep. on **64-bit processors**
(note: not using full bitwidth!)

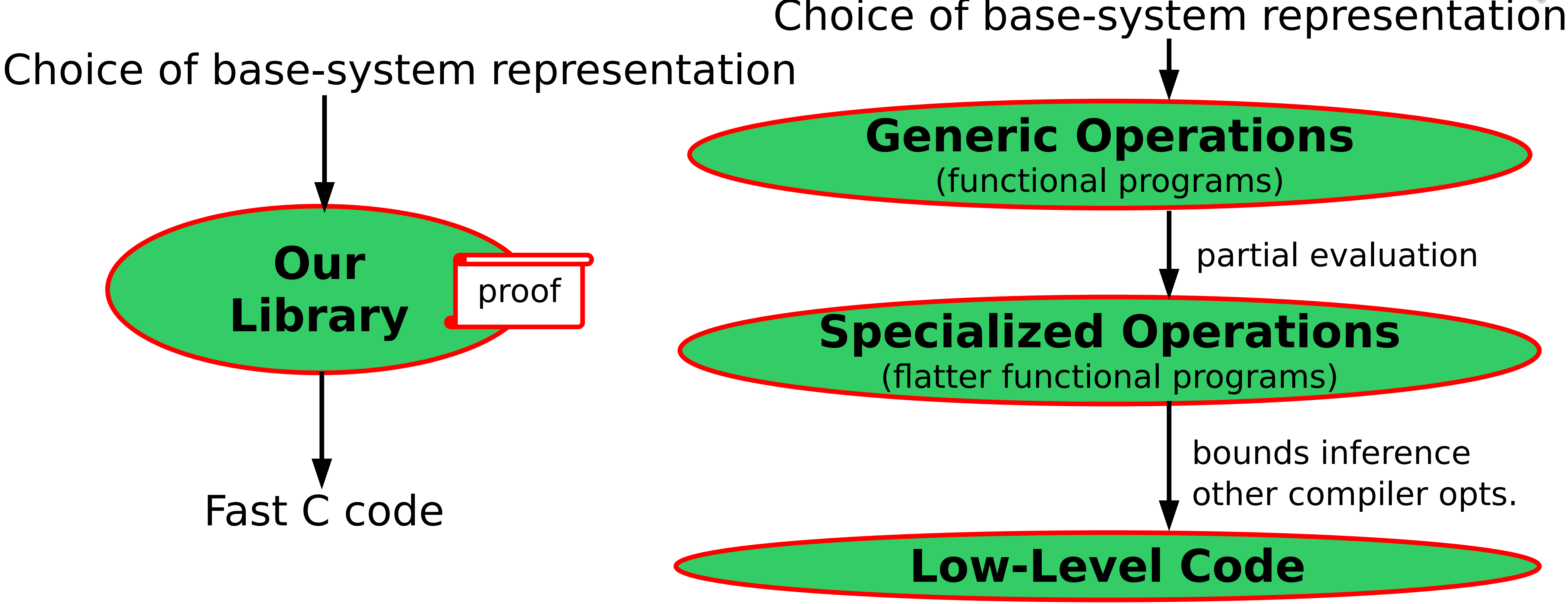
Also.... $25.5 \times 2 = 51$.

$t = s_0 + 2^{25.5}\ s_1 + 2^{2 \times 25.5}\ s_2 + 2^{3 \times 25.5}\ s_3 + \ldots$

$t = s_0 + 2^{26}\ s_1 + 2^{51}\ s_2 + 2^{77}\ s_3 + \ldots$

champion rep. on 32**-bit processors**
(note: nonuniform bitwidths!)

The Basic Idea



Example: Multiplication (for modulus 2^{127} - 1)

$$s = s_0 + 2^{43} s_1 + 2^{85} s_2$$

$$t = t_0 + 2^{43} t_1 + 2^{85} t_2$$

$$s \times t = u = u_0\ u_1\ u_2\ u_3\ u_4$$

$$\begin{aligned} s \times t = 1 \times s_0t_0 &+ 2^{43} \times s_0t_1 + 2^{85} \times s_0t_2 \\ &+ 2^{43} \times s_1t_0 + 2^{86} \times s_1t_1 + 2^{128} \times s_1t_2 \\ &+ 2^{85} \times s_2t_0 + 2^{128} \times s_2t_1 + 2^{170} \times s_2t_2 \end{aligned}$$

$$\boxed{u_0 = s_0t_0}$$

$$\boxed{u_1 = s_0t_1 + s_1t_0}$$

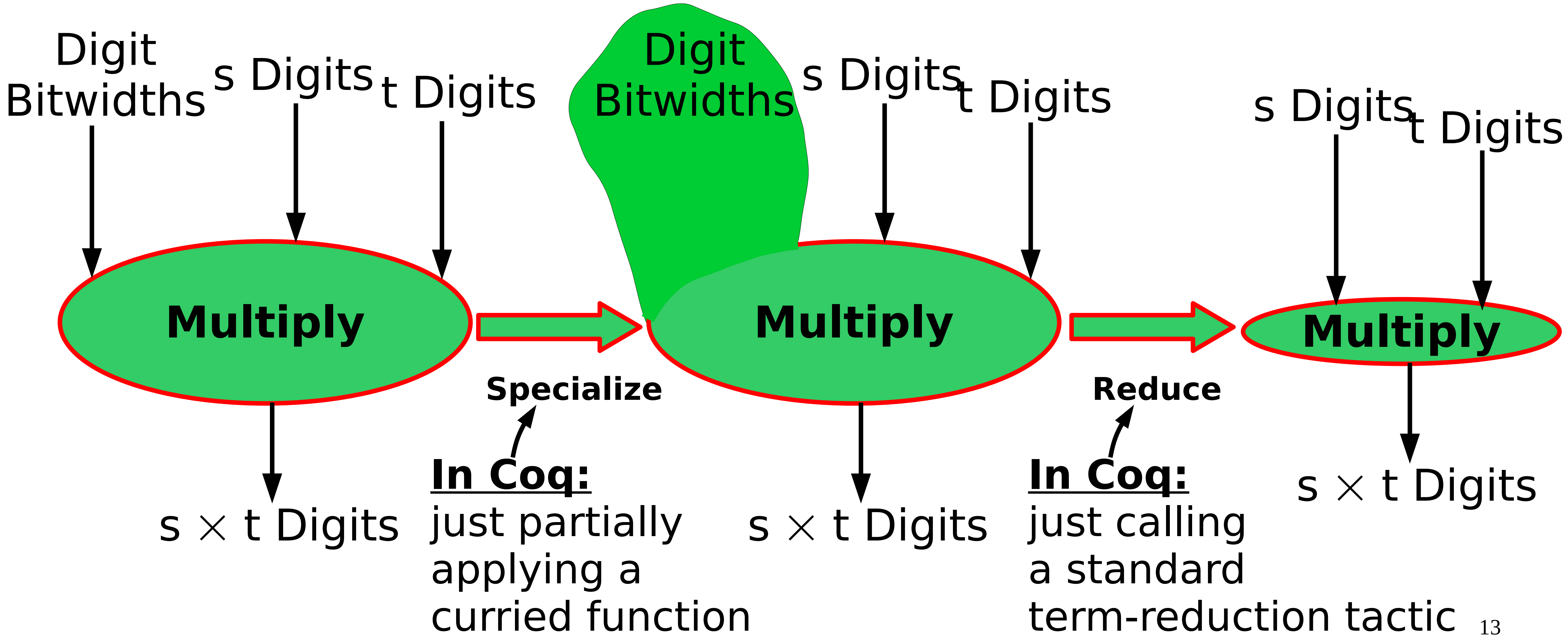
$$\boxed{u_2 = s_0t_2 + 2s_1t_1 + s_2t_0}$$

$$\boxed{u_3 = 2s_1t_2 + 2s_2t_1}$$

$$\boxed{u_4 = s_2t_2}$$

$$\begin{aligned} u &= u_0 + 2^{43} u_1 + 2^{85} u_2 + 2^{127} (u_3 + 2^{43} u_4) \\ &= (u_0 + u_3) + 2^{43} (u_1 + u_4) + 2^{85} u_2 \end{aligned}$$

Time for Some Partial Evaluation



An Example

```
Definition w (i:nat) : Z := 2^Qceiling((25+1/2)*i).

Example base_25_5_mul (f g:tuple Z 10) :
  { fg : tuple Z 10 |
   (eval w fg) mod (2^255-19)
   = (eval w f * eval w g) mod (2^255-19) }.
```

```
(f0*g9+f1*g8+f2*g7+f3*g6+f4*g5+f5*g4+f6*g3+f7*g2+f8*g1+f9*g0,
 f0*g8+2*f1*g7+f2*g6+2*f3*g5+f4*g4+2*f5*g3+f6*g2+2*f7*g1+f8*g0+38*f9*g9,
 f0*g7+f1*g6+f2*g5+f3*g4+f4*g3+f5*g2+f6*g1+f7*g0+19*f8*g9+19*f9*g8,
 f0*g6+2*f1*g5+f2*g4+2*f3*g3+f4*g2+2*f5*g1+f6*g0+38*f7*g9+19*f8*g8+38*f9*g7,
 f0*g5+f1*g4+f2*g3+f3*g2+f4*g1+f5*g0+19*f6*g9+19*f7*g8+19*f8*g7+19*f9*g6,
 f0*g4+2*f1*g3+f2*g2+2*f3*g1+f4*g0+38*f5*g9+19*f6*g8+38*f7*g7+19*f8*g6+38*f9*g5,
 f0*g3+f1*g2+f2*g1+f3*g0+19*f4*g9+19*f5*g8+19*f6*g7+19*f7*g6+19*f8*g5+19*f9*g4,
 f0*g2+2*f1*g1+f2*g0+38*f3*g9+19*f4*g8+38*f5*g7+19*f6*g6+38*f7*g5+19*f8*g4+38*f9*g3,
 f0*g1+f1*g0+19*f2*g9+19*f3*g8+19*f4*g7+19*f5*g6+19*f6*g5+19*f7*g4+19*f8*g3+19*f9*g2,
 f0*g0+38*f1*g9+19*f2*g8+38*f3*g7+19*f4*g6+38*f5*g5+19*f6*g4+38*f7*g3+19*f8*g2+38*f9*g1)
```

Compiling to Low-Level Code

$$1 \times (1 \times 2^{52} + (1 \times x + 0)) + (1 \times (1 \times (-y) + 0) + 0)$$

reify to syntax tree

constant-fold

$$(2^{52} + x) - y$$

flatten

let $c = 2^{52} + x$ in
let $d = c - y$ in
d

Assume: $0 \leq x, y \leq 2^{51} + 2^{48}$
Deduce: $2^{52} \leq c \leq 2^{52} + 2^{51} + 2^{48}$
Deduce: $2^{51} - 2^{48} \leq d \leq 2^{52} + 2^{51} + 2^{48}$

infer bounds

```
uint64_t c = 2^52 + x;
uint64_t d = c - y;
return d
```

Implementation and Experiments

- ~38 kloc in full library (including significant parts that belong in stdlib)
- Very little code needed to instantiate to new prime moduli.
- In fact, we wrote a Python script (under 3000 lines) to generate parameters automatically from prime numbers, written suggestively, e.g. $2^{256} - 2^{224} + 2^{192} + 2^{96} – 1$.
- This script is outside the TCB, since any successful compilation is guaranteed to implement correct arithmetic.

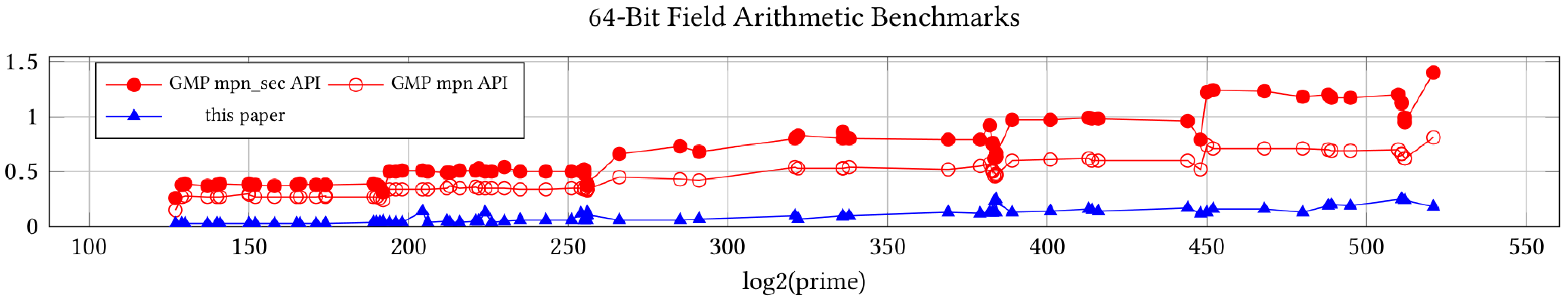
Q: Where do we get a lot of reasonable moduli?

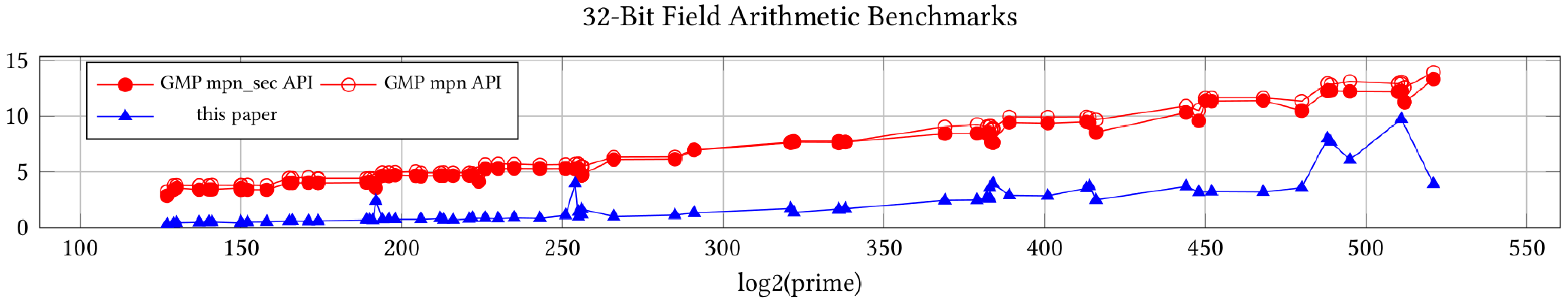
A: Scrape all prime numbers appearing in a popular mailing list.

We used the elliptic curves list at moderncrypto.org.
We found about 80 primes.

Only a few turned out to be terrible ideas posted by newbies.

Many-Primes Experiment





P256 Mixed Addition

Implementation	CPU cycles	μs at 2.6GHz
OpenSSL AMD64+ADX asm	544	.21
OpenSSL AMD64 asm	644	.25
this work, icc	1112	.43
this work, gcc	1808	.70
OpenSSL C	1968	.76

Next Steps

- Close the performance & trust gap with assembly by extending verified pipeline.
  - Maybe verify an *equivalence checker* between our code and code handwritten by experts.
- Extend code generation to apply to higher-level crypto code, like curve arithmetic, not just field arithmetic.
  - Requires handling loops, function calls, etc.
- Goal (medium-term): a complete TLS implementation derived in a correct-by-construction way!

https://github.com/mit-plv/fiat-crypto