A functional tour of automatic differentiation with Racket

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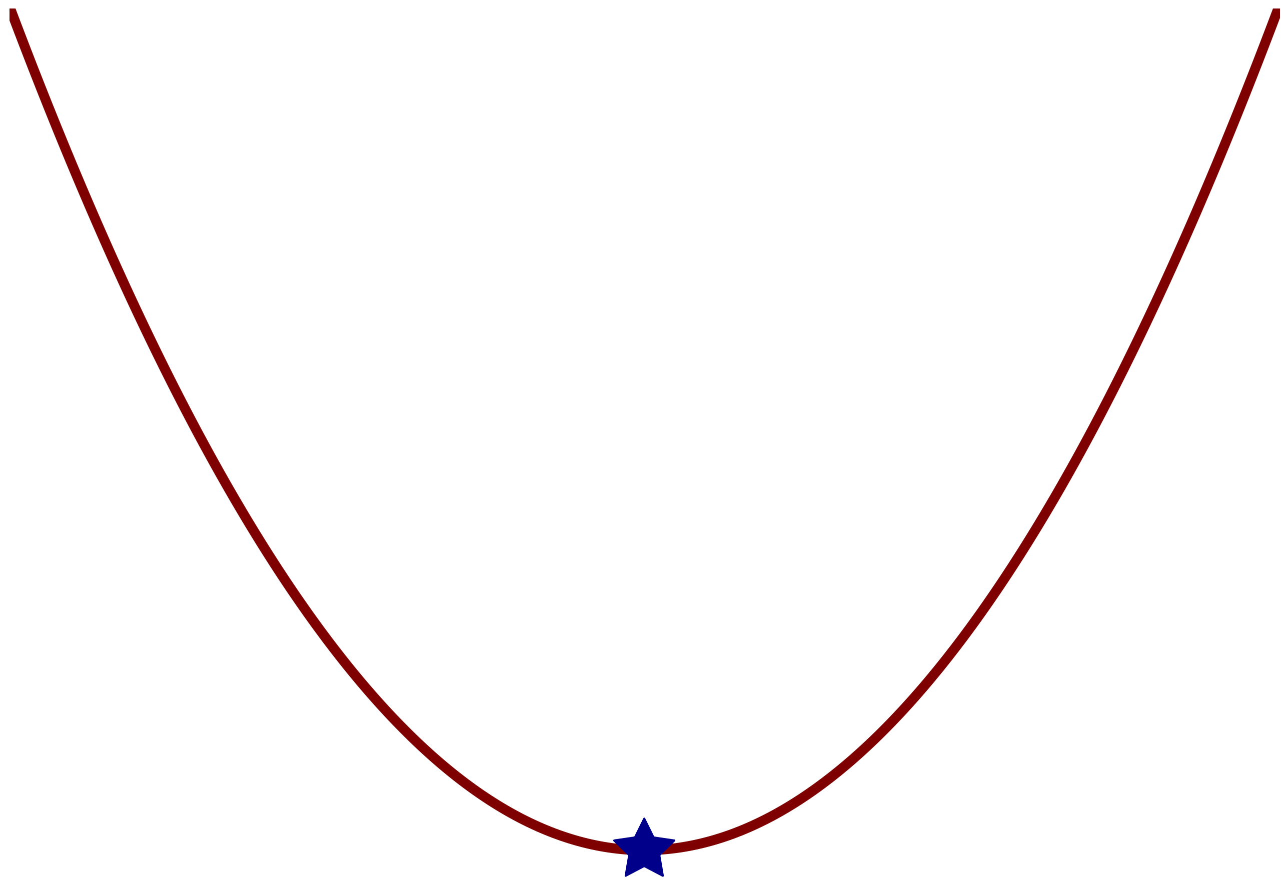
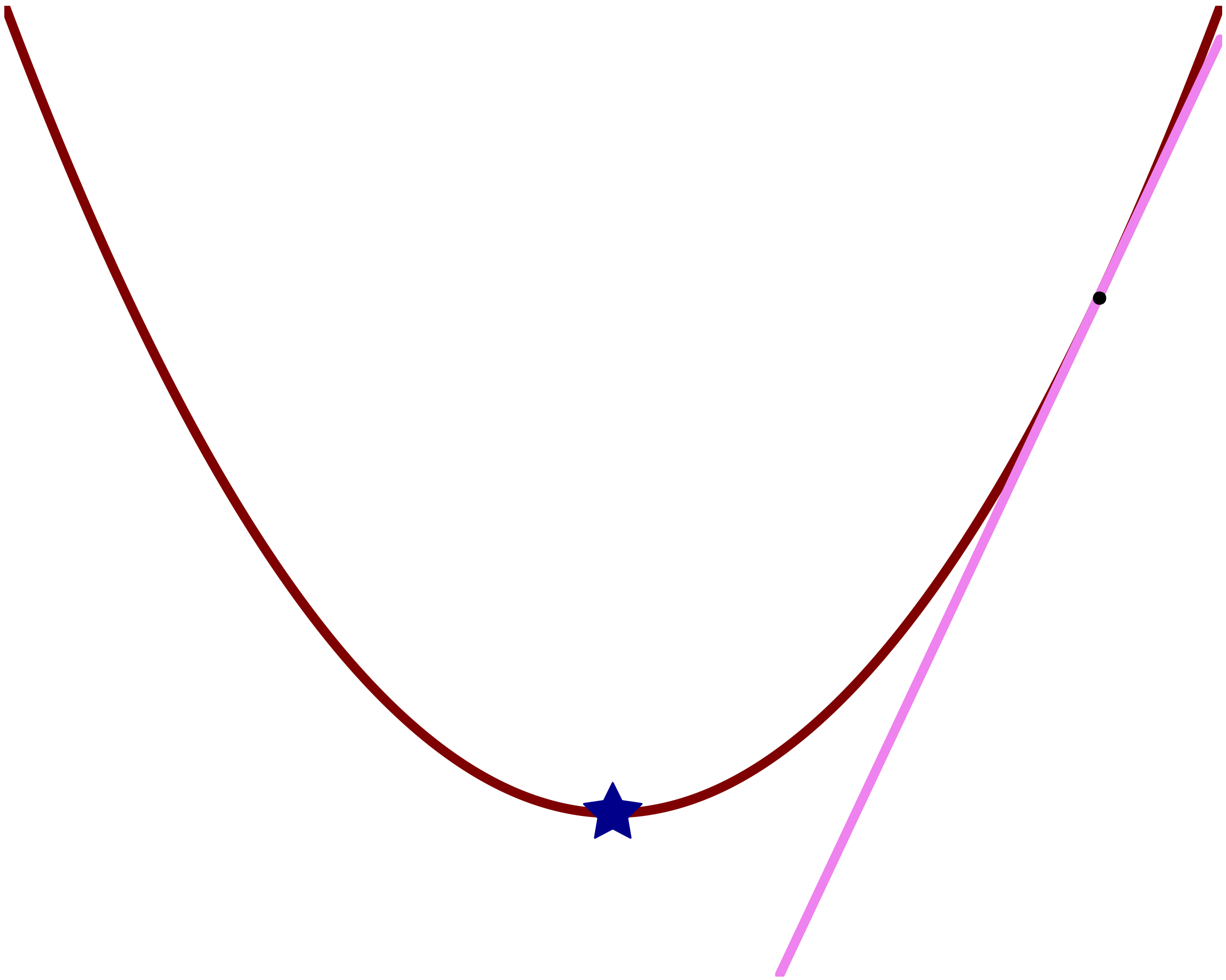
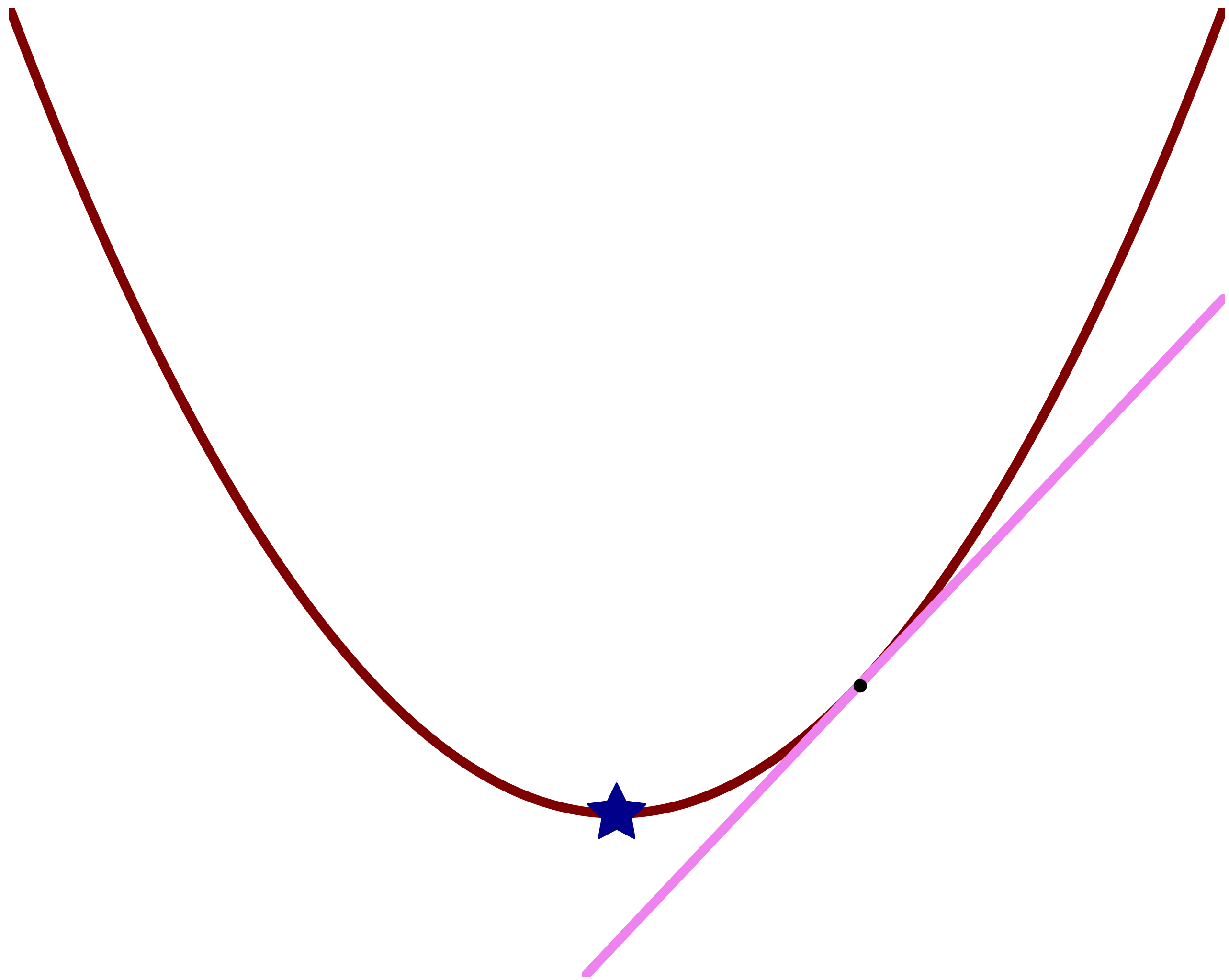
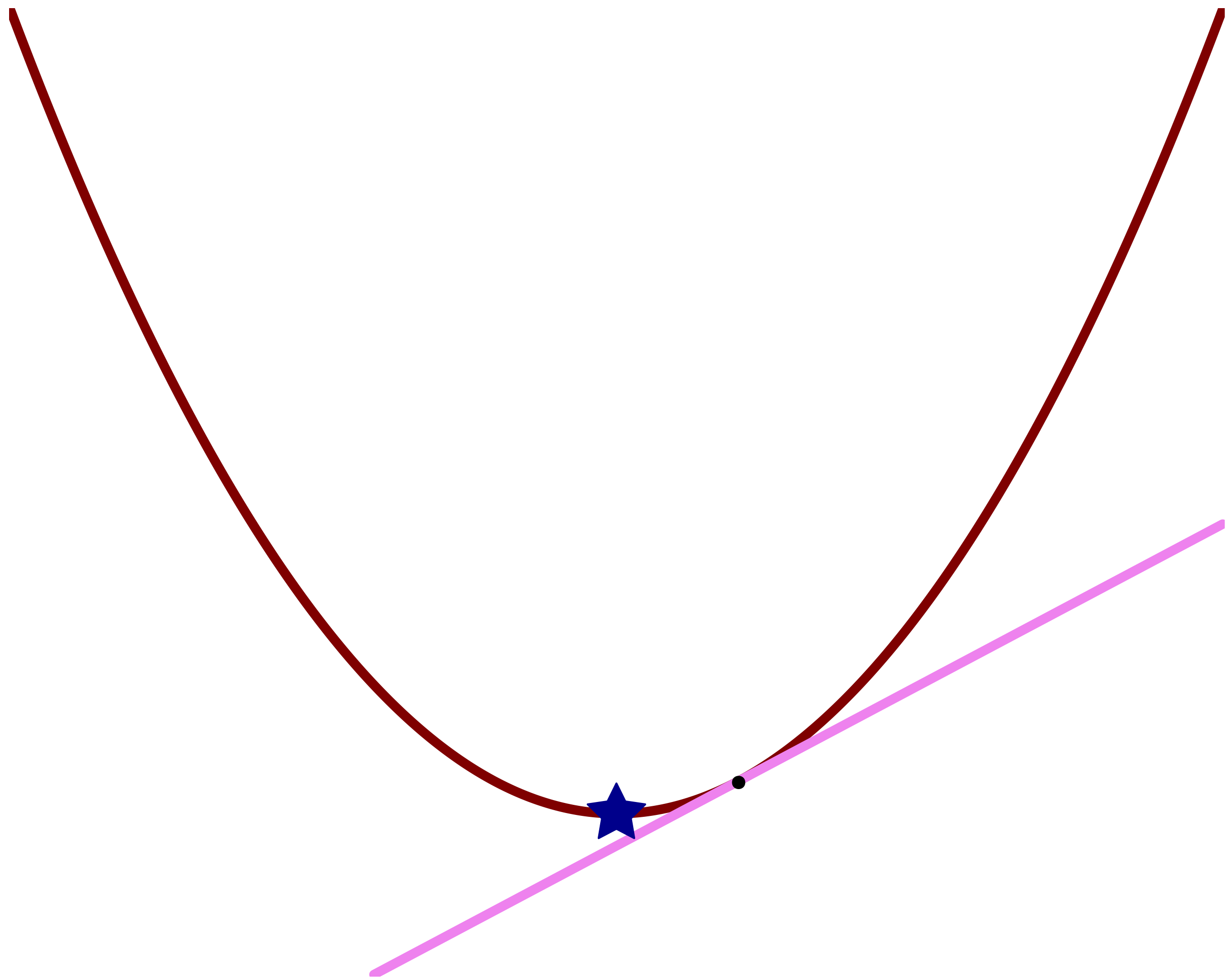
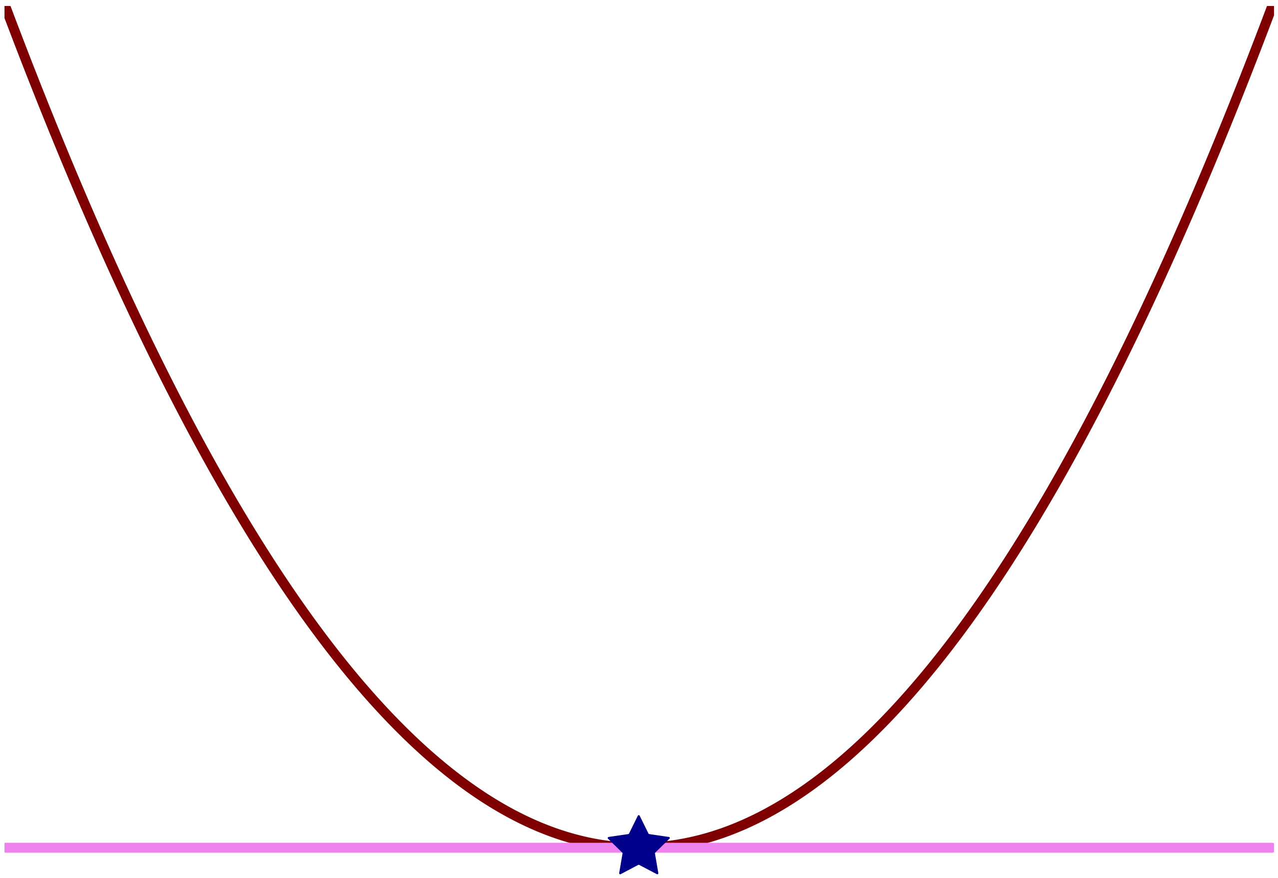










minimize f

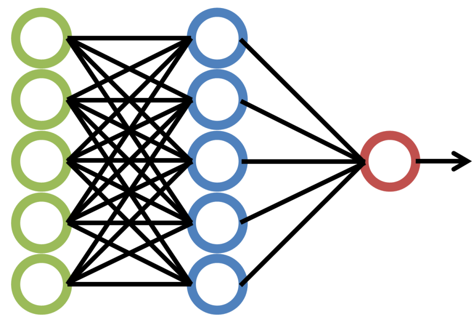
minimize f Df

$$f(x) = x^2$$

$$f(x) = x^2$$
$$Df(x) = ??$$

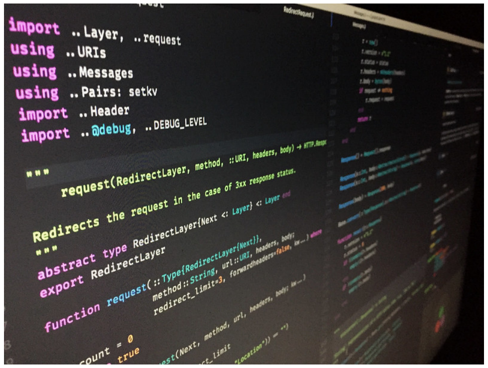
$$f(x) = x^2$$
$$Df(x) = 2x$$

$$f(x) =$$



?

$$f(x) =$$


import ..Layer, ..request
using ..URIs
using ..Messages
using ..Pairs: setkv
import ..Header
import ..@debug, ..DEBUG_LEVEL
Redirects the request in the case of 3xx response status.
abstract type RedirectLayer
export RedirectLayer

?

$$Df(x) \approx \frac{f(x+h) - f(x)}{h}$$

Symbolic?

Automatic

Overview

- Some syntax
- Differentiation
- Automatic differentiation algorithm(s)
- Implementation

```
(cons 'a 'b) => (a . b)
```

```
(cons 'a 'b) => (a . b)
(cons 'a (cons 'b 'c)) => (a b . c)
(cons 'a (cons 'b null)) => (a b)
```

```
(cons 'a 'b) => (a . b)
(cons 'a (cons 'b 'c)) => (a b . c)
(cons 'a (cons 'b null)) => (a b)
(list 'a 'b) => (a b)
```

```
(cons 'a 'b) => (a . b)

(cons 'a (cons 'b 'c)) => (a b . c)

(cons 'a (cons 'b null)) => (a b)

(list 'a 'b) => (a b)

(3 1 ((4 1)) (5 . 9) 3)
```

```
(cons 'a 'b) => (a . b)
(car '(a . b)) => a
(cdr '(a . b)) => b
```

```
(car '(a b c)) => a
(cdr '(a b c)) => (b c)
```

```
(define (multiply x y) (* x y))
```

```
(define ((multiply x) y) (* x y))
```

```
(define (sum . xs) (apply + xs))
```

Differentiation

Differentiation

The best linear approximation to a function about a point (if it exists)

Differentiation

The best linear approximation to a function about a point (if it exists)

Function f or `f`

Derivative Df or `(D f)`

Differentiation

function $f(x)$

find a with

$$f(x) - f(x_0) \approx a\,(x - x_0)$$

Differentiation

function $f(x)$

find a with

$$f(x) - f(x_0) \approx a\,(x - x_0)$$

$$f(x) - f(x_0) = a\,(x - x_0) + O((x - x_0)^2)$$

Differentiation

function $f(x)$

find a with

$$f(x) - f(x_0) \approx a\,(x - x_0)$$

$$f(x) - f(x_0) = a\,(x - x_0) + O((x - x_0)^2)$$

$$f(x) - f(x_0) = Df(x_0)\,(x - x_0) + O((x - x_0)^2)$$

Differentiation

function $f(x, y)$

find a, b with

$$f(x, y) - f(x_0, y_0) \approx a\,(x - x_0) + b\,(y - y_0)$$

Differentiation

function $f(x, y)$

find a, b with

$$f(x, y) - f(x_0, y_0) \approx a\,(x - x_0) + b\,(y - y_0)$$

$$f(x, y) - f(x_0, y_0) \approx D_0 f(x_0, y_0)\,(x - x_0) + D_1 f(x_0, y_0)\,(y - y_0)$$

Differentiation

function $f(x, y)$

find a, b with

$$f(x, y) - f(x_0, y_0) \approx a\,(x - x_0) + b\,(y - y_0)$$

$$f(x, y) - f(x_0, y_0) \approx D_0 f(x_0, y_0)\,(x - x_0) + D_1 f(x_0, y_0)\,(y - y_0)$$

Partial derivative $D_i f$ or `(partial i f)`

Differentiation

function $f(x, y)$

find a, b with

$$f(x, y) - f(x_0, y_0) \approx a\,(x - x_0) + b\,(y - y_0)$$

$$f(x, y) - f(x_0, y_0) \approx D_0 f(x_0, y_0)\,(x - x_0) + D_1 f(x_0, y_0)\,(y - y_0)$$

Partial derivative $D_i f$ or `(partial i f)`

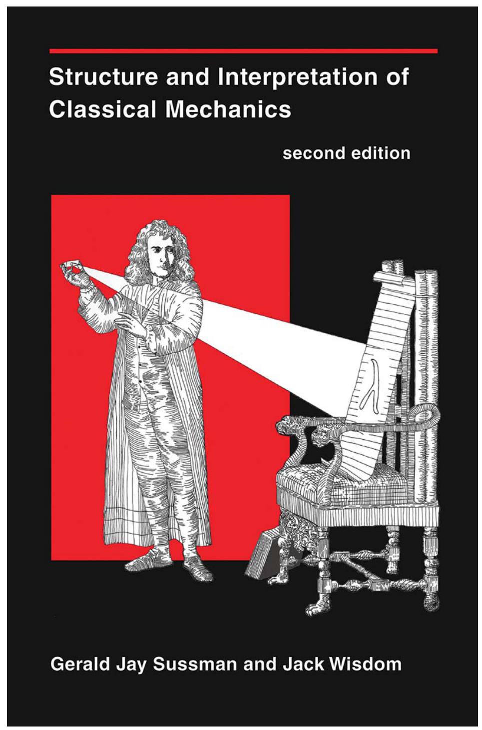
$$Df(x, y) = (D_0 f(x, y), D_1 f(x, y))$$

BOOK

Structure and Interpretation of Classical Mechanics (2nd ed.)

`https://mitpress.mit.edu/sites/default/files/titles/content/sicm_edition_2/book.html`

Gerald Jay Sussman & Jack Wisdom (2015)



Composition

$$f(x) = g(h(x))$$

$$Df(x) = Dg(f(x)) \cdot Df(x)$$

Composition

$$f(x, y) = g(u(x, y), v(x, y))$$

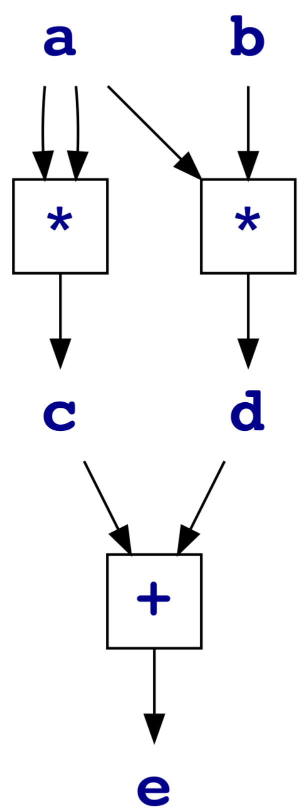
$$\begin{aligned} Df(x, y) &= D_0 g(u(x, y), v(x, y)) \cdot Du(x, y) \\ &+ D_1 g(u(x, y), v(x, y)) \cdot Dv(x, y) \end{aligned}$$

Arithmetic expressions

```
(+ (* a a) (* a b))
```

Arithmetic expressions

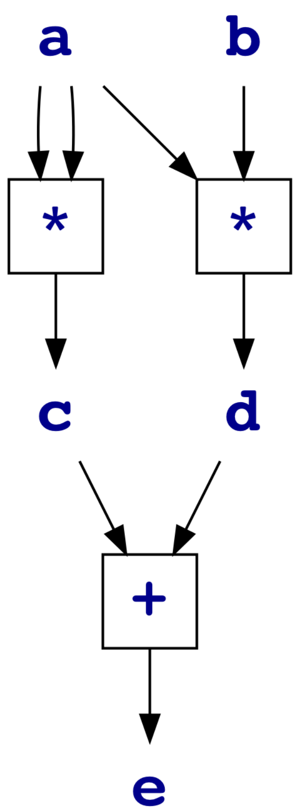
`(+ (* a a) (* a b))`



c ← (* a a)
d ← (* a b)
e ← (+ c d)

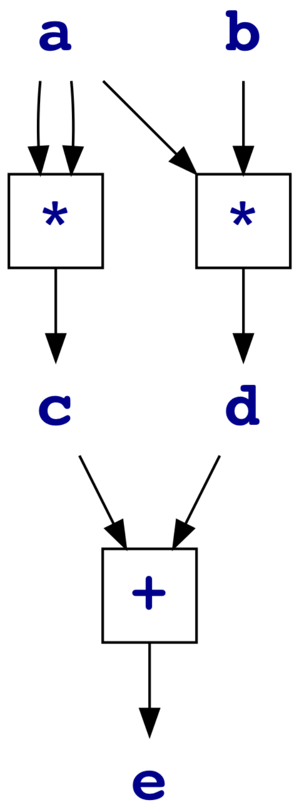
Automatic differentiation

Compute $\mathrm{D}f(a, b)$



Automatic differentiation

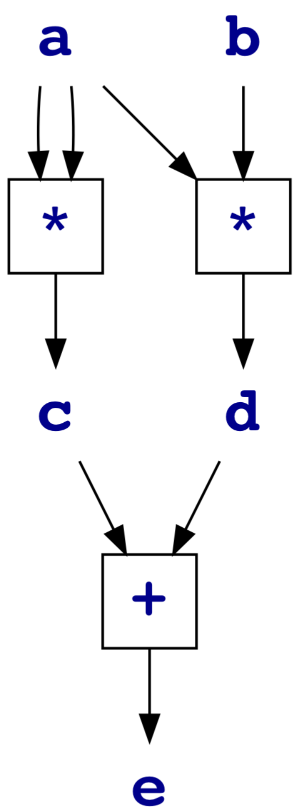
Compute $\mathrm{D}f(a, b)$



$$\frac{da}{dr} = 1$$
$$\frac{db}{dr} = 0$$

Automatic differentiation

Compute $Df(a, b)$



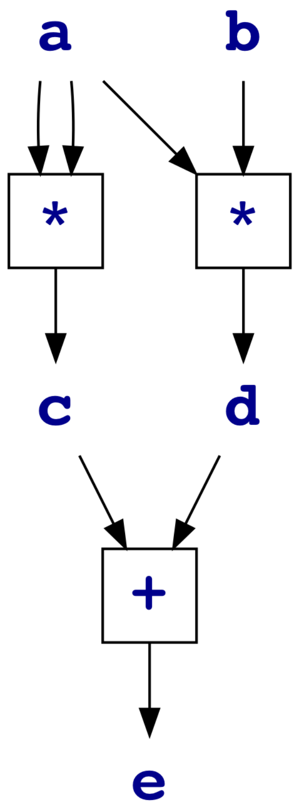
$$\frac{da}{dr} = 1$$

$$\frac{db}{dr} = 0$$

$$\frac{dc}{dr} = D_0(*)(a, a)\frac{da}{dr} + D_1(*)(a, a)\frac{da}{dr}$$

Automatic differentiation

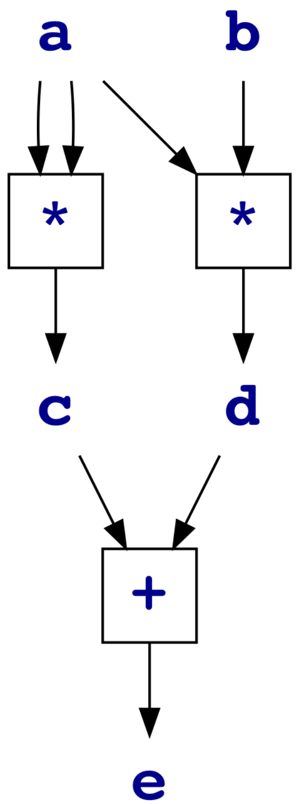
Compute $\mathrm{D}f(a, b)$



$$\frac{da}{dr} = 1$$
$$\frac{db}{dr} = 0$$
$$\frac{dc}{dr} = D_0(*)(a, a)\frac{da}{dr} + D_1(*)(a, a)\frac{da}{dr}$$
$$\frac{dd}{dr} = D_0(*)(a, b)\frac{da}{dr} + D_1(*)(a, b)\frac{db}{dr}$$

Automatic differentiation

Compute $Df(a, b)$



$$\frac{da}{dr} = 1$$

$$\frac{db}{dr} = 0$$

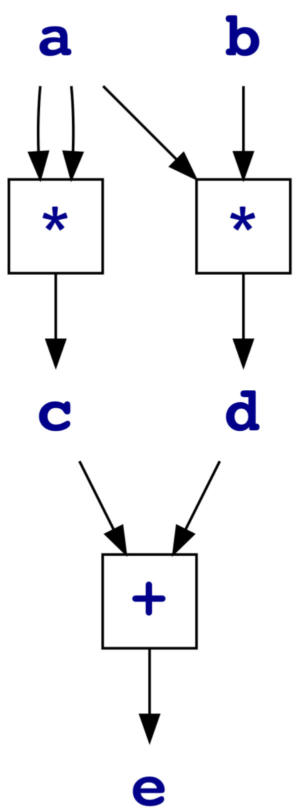
$$\frac{dc}{dr} = D_0(*)(a, a)\frac{da}{dr} + D_1(*)(a, a)\frac{da}{dr}$$

$$\frac{dd}{dr} = D_0(*)(a, b)\frac{da}{dr} + D_1(*)(a, b)\frac{db}{dr}$$

$$\frac{de}{dr} = D_0(+)(c, d)\frac{dc}{dr} + D_1(+)(c, d)\frac{dd}{dr}$$

Automatic differentiation

Compute $Df(a, b)$



$$\frac{da}{dr} = 1$$

$$\frac{db}{dr} = 0$$

$$\frac{dc}{dr} = D_0(*)(a, a)\frac{da}{dr} + D_1(*)(a, a)\frac{da}{dr}$$

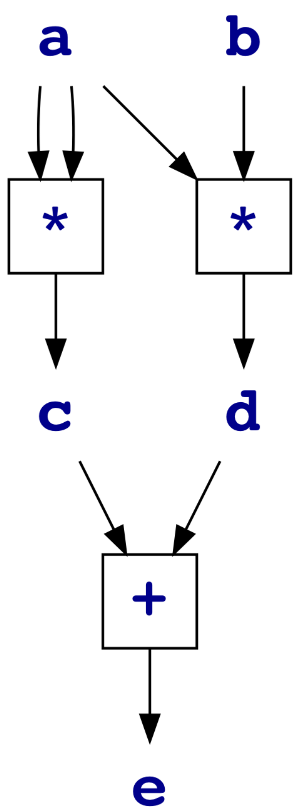
$$\frac{dd}{dr} = D_0(*)(a, b)\frac{da}{dr} + D_1(*)(a, b)\frac{db}{dr}$$

$$\frac{de}{dr} = D_0(+)(c, d)\frac{dc}{dr} + D_1(+)(c, d)\frac{dd}{dr}$$

$$D_0 f(a, b) = \frac{de}{dr}$$

Automatic differentiation

Compute $\mathrm{D}f(a, b)$



$$\frac{da}{dr} = 1 \qquad \frac{db}{dr} = 0$$

$$\frac{dc}{dr} = \mathrm{D}_0(*)(a, a)\frac{da}{dr} + \mathrm{D}_1(*)(a, a)\frac{da}{dr}$$

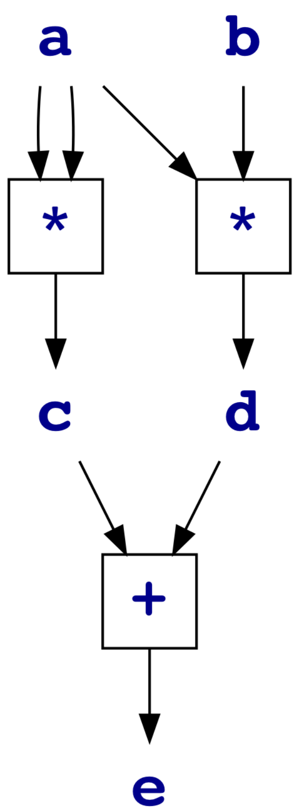
$$\frac{dd}{dr} = \mathrm{D}_0(*)(a, b)\frac{da}{dr} + \mathrm{D}_1(*)(a, b)\frac{db}{dr}$$

$$\frac{de}{dr} = \mathrm{D}_0(+)(c, d)\frac{dc}{dr} + \mathrm{D}_1(+)(c, d)\frac{dd}{dr}$$

$$\mathrm{D}_0 f(a, b) = \frac{de}{dr}$$

Automatic differentiation

Compute $Df(a, b)$



$$\boxed{\begin{aligned}\frac{da}{dr} &= 0\\ \frac{db}{dr} &= 1\end{aligned}}$$

$$\frac{dc}{dr} = D_0(*)(a, a)\frac{da}{dr} + D_1(*)(a, a)\frac{da}{dr}$$

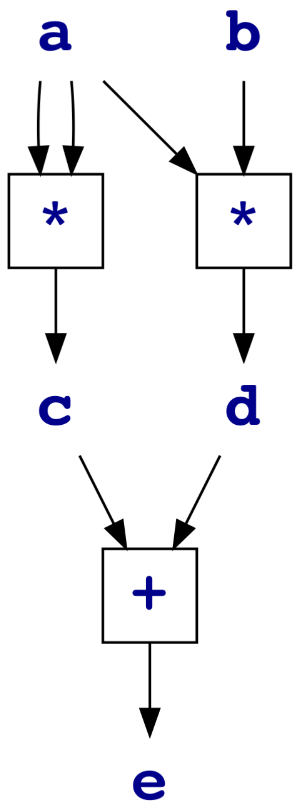
$$\frac{dd}{dr} = D_0(*)(a, b)\frac{da}{dr} + D_1(*)(a, b)\frac{db}{dr}$$

$$\frac{de}{dr} = D_0(+)(c, d)\frac{dc}{dr} + D_1(+)(c, d)\frac{dd}{dr}$$

$$D_1 f(a, b) = \frac{de}{dr}$$

Automatic differentiation

Compute $Df(a, b)$

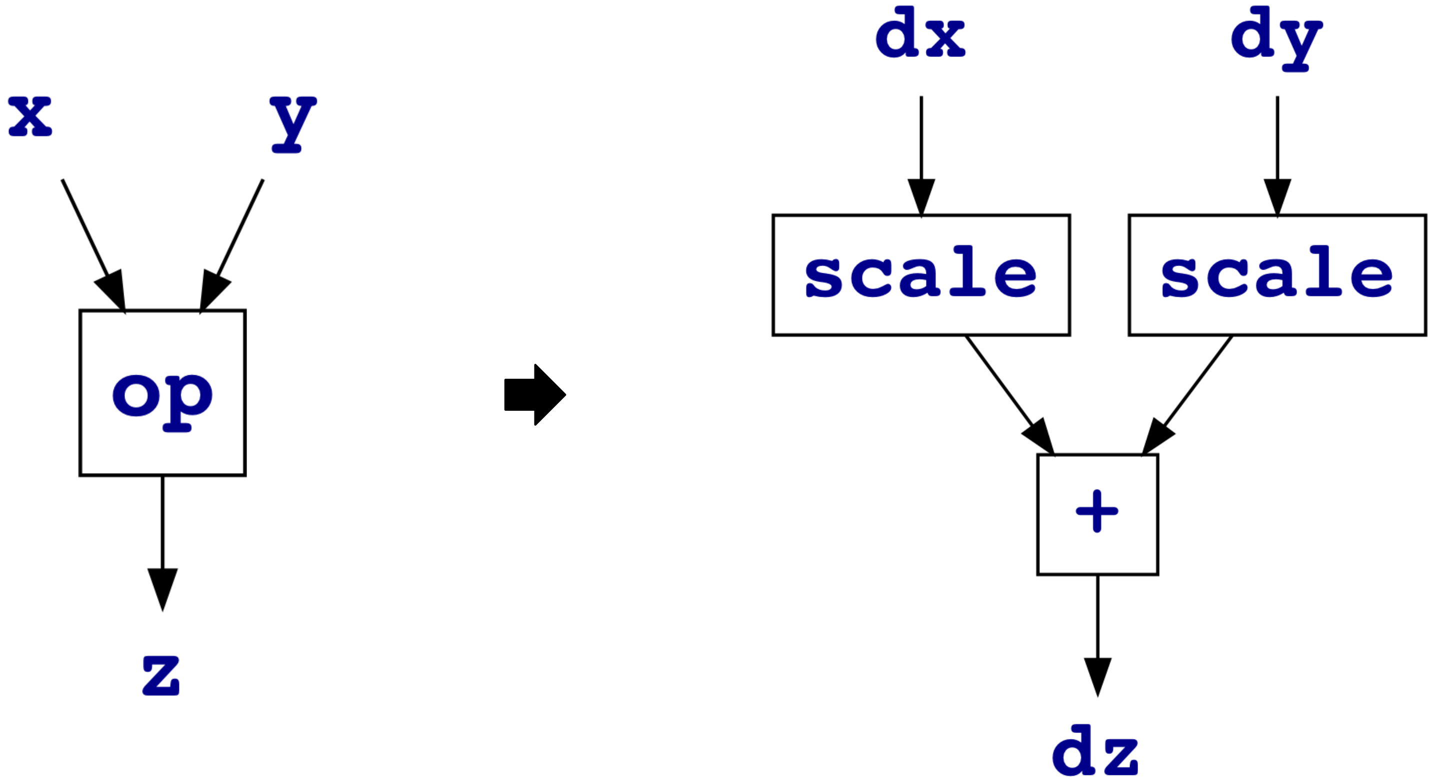


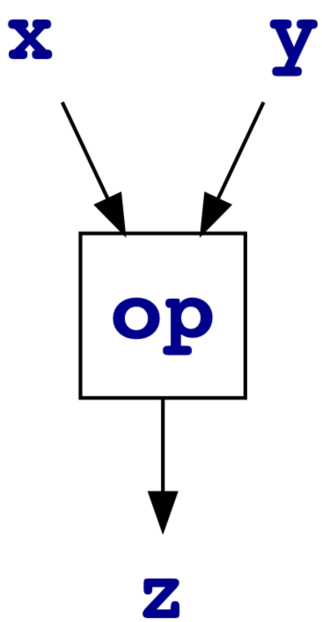
$$\frac{da}{dr} = 0$$
$$\frac{db}{dr} = 1$$
$$\frac{dc}{dr} = D_0(*)(a, a)\frac{da}{dr} + D_1(*)(a, a)\frac{da}{dr}$$
$$\frac{dd}{dr} = D_0(*)(a, b)\frac{da}{dr} + D_1(*)(a, b)\frac{db}{dr}$$
$$\frac{de}{dr} = D_0(+)(c, d)\frac{dc}{dr} + D_1(+)(c, d)\frac{dd}{dr}$$
$$D_1 f(a, b) = \frac{de}{dr}$$

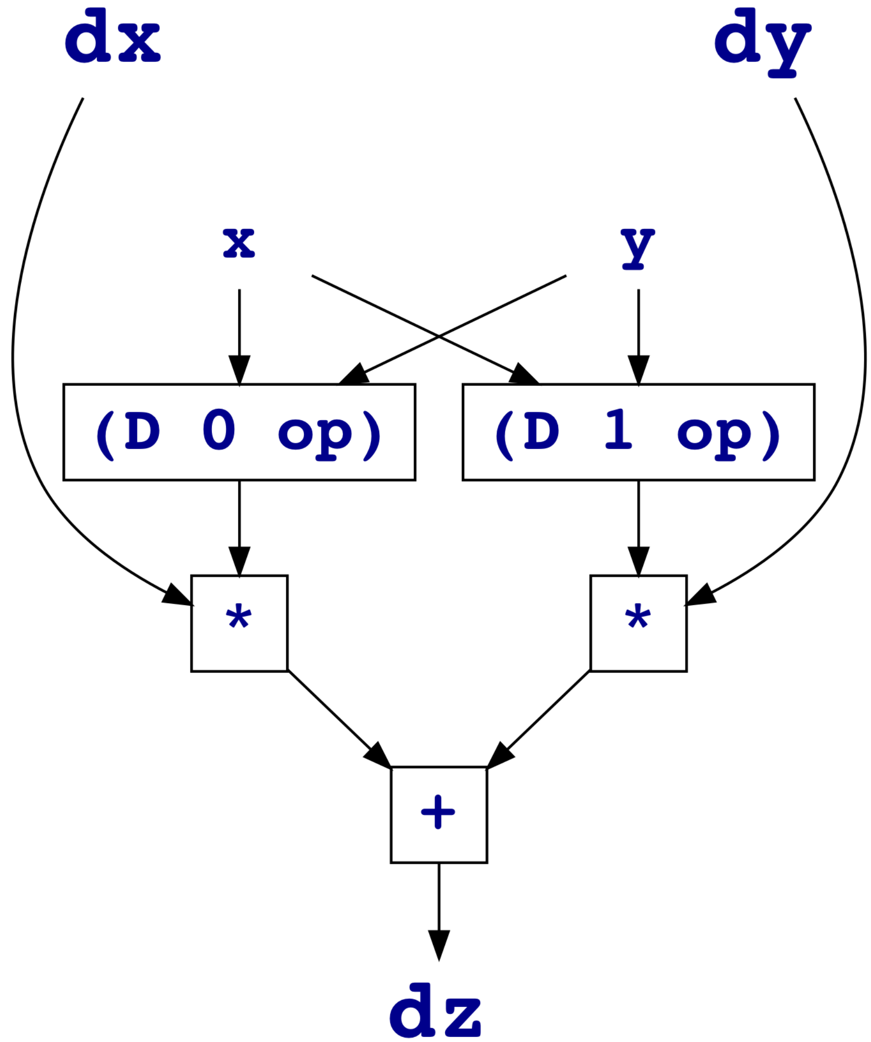
Forward mode

Will write **dx** instead of $\frac{dx}{dr}$

Known as *perturbation variables*

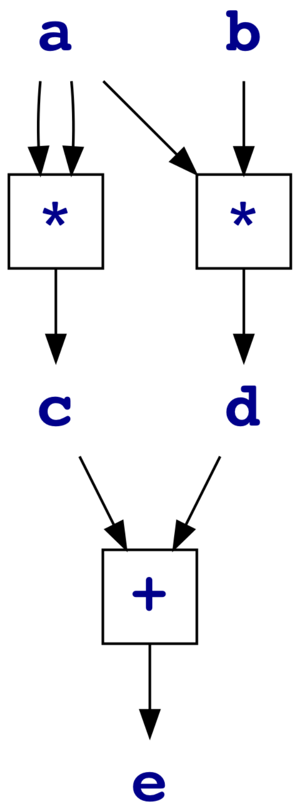

x
y
op
z
dx
dy
scale
scale
+
dz


x
y
op
z


dx
dy
x
y
(D 0 op)
(D 1 op)
*
*
+
dz

Automatic differentiation

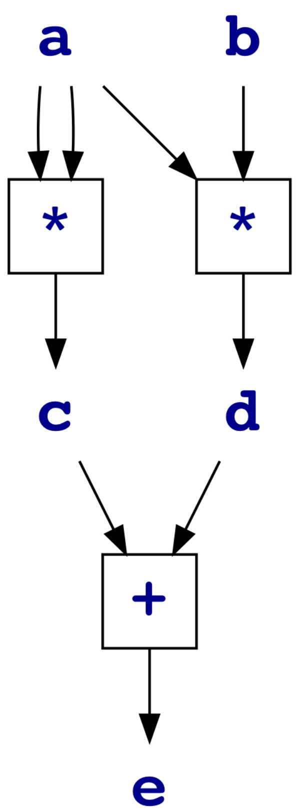
Compute $\mathrm{D}f(a, b)$



Automatic differentiation

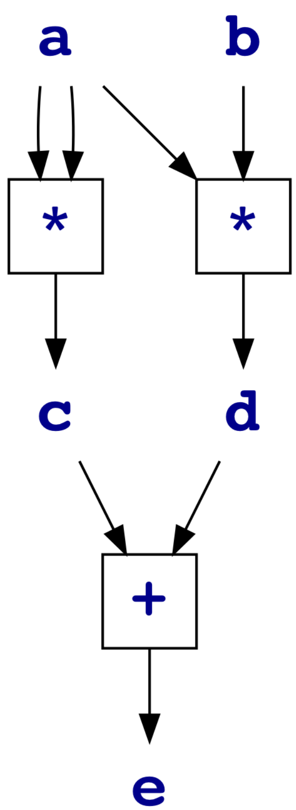
Compute $\mathrm{D}f(a, b)$

$\frac{ds}{de} = 1$



Automatic differentiation

Compute $\mathrm{D}f(a, b)$

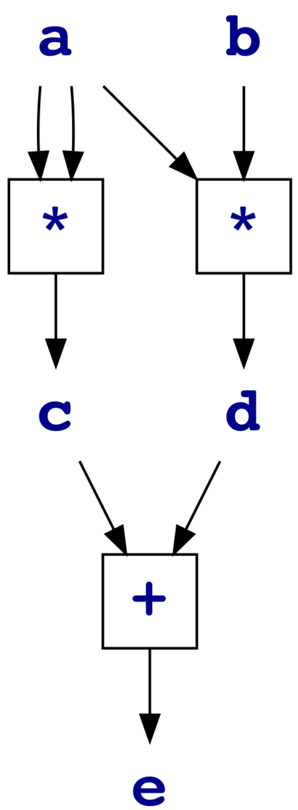


$$\frac{ds}{de} = 1$$

$$\frac{ds}{dd} = \mathrm{D}_1(+)(c, d)\frac{ds}{de}$$

Automatic differentiation

Compute $Df(a, b)$



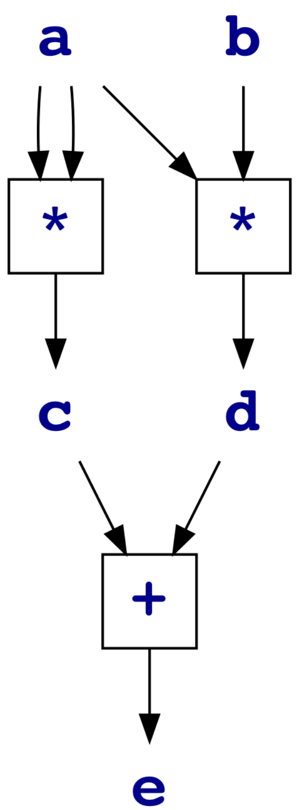
$$\frac{ds}{de} = 1$$

$$\frac{ds}{dd} = D_1(+)(c, d)\frac{ds}{de}$$

$$\frac{ds}{dc} = D_0(+)(c, d)\frac{ds}{de}$$

Automatic differentiation

Compute $\mathrm{D}f(a, b)$



$$\frac{ds}{de} = 1$$

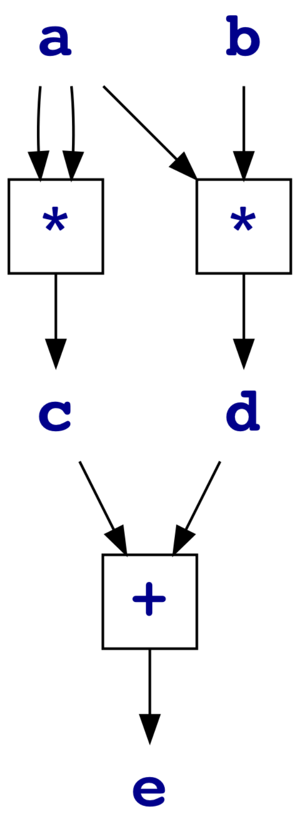
$$\frac{ds}{dd} = D_1(+)(c, d)\frac{ds}{de}$$

$$\frac{ds}{dc} = D_0(+)(c, d)\frac{ds}{de}$$

$$\frac{ds}{db} = D_1(*)(a, b)\frac{ds}{dd}$$

Automatic differentiation

Compute $\mathrm{D}f(a, b)$



$$\frac{ds}{de} = 1$$

$$\frac{ds}{dd} = \mathrm{D}_1(+)(c, d)\frac{ds}{de}$$

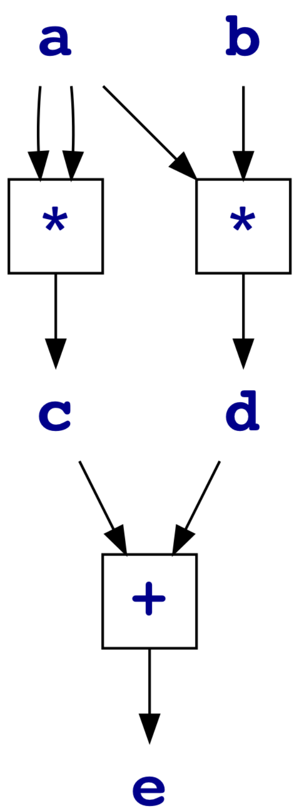
$$\frac{ds}{dc} = \mathrm{D}_0(+)(c, d)\frac{ds}{de}$$

$$\frac{ds}{db} = \mathrm{D}_1(*)(a, b)\frac{ds}{dd}$$

$$\frac{ds}{da} = \mathrm{D}_0(*)(a,a)\frac{ds}{dc} + \mathrm{D}_1(*)(a, a)\frac{ds}{dc} + \mathrm{D}_0(*)(a, b)\frac{ds}{dd}$$

Automatic differentiation

Compute $Df(a, b)$



$$\frac{ds}{de} = 1$$

$$\frac{ds}{dd} = D_1(+)(c, d)\frac{ds}{de}$$

$$\frac{ds}{dc} = D_0(+)(c, d)\frac{ds}{de}$$

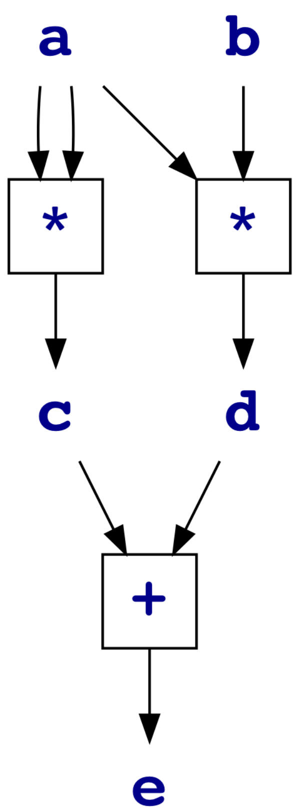
$$\frac{ds}{db} = D_1(*)(a, b)\frac{ds}{dd}$$

$$\frac{ds}{da} = D_0(*)(a, a)\frac{ds}{dc} + D_1(*)(a, a)\frac{ds}{dc} + D_0(*)(a, b)\frac{ds}{dd}$$

$$Df(a, b) = \left(\frac{ds}{da}, \frac{ds}{db}\right)$$

Automatic differentiation

Compute $Df(a, b)$



$$\frac{ds}{de} = 1$$

$$\frac{ds}{dd} = D_1(+)(c, d)\frac{ds}{de}$$

$$\frac{ds}{dc} = D_0(+)(c, d)\frac{ds}{de}$$

$$\frac{ds}{db} = D_1(*)(a, b)\frac{ds}{dd}$$

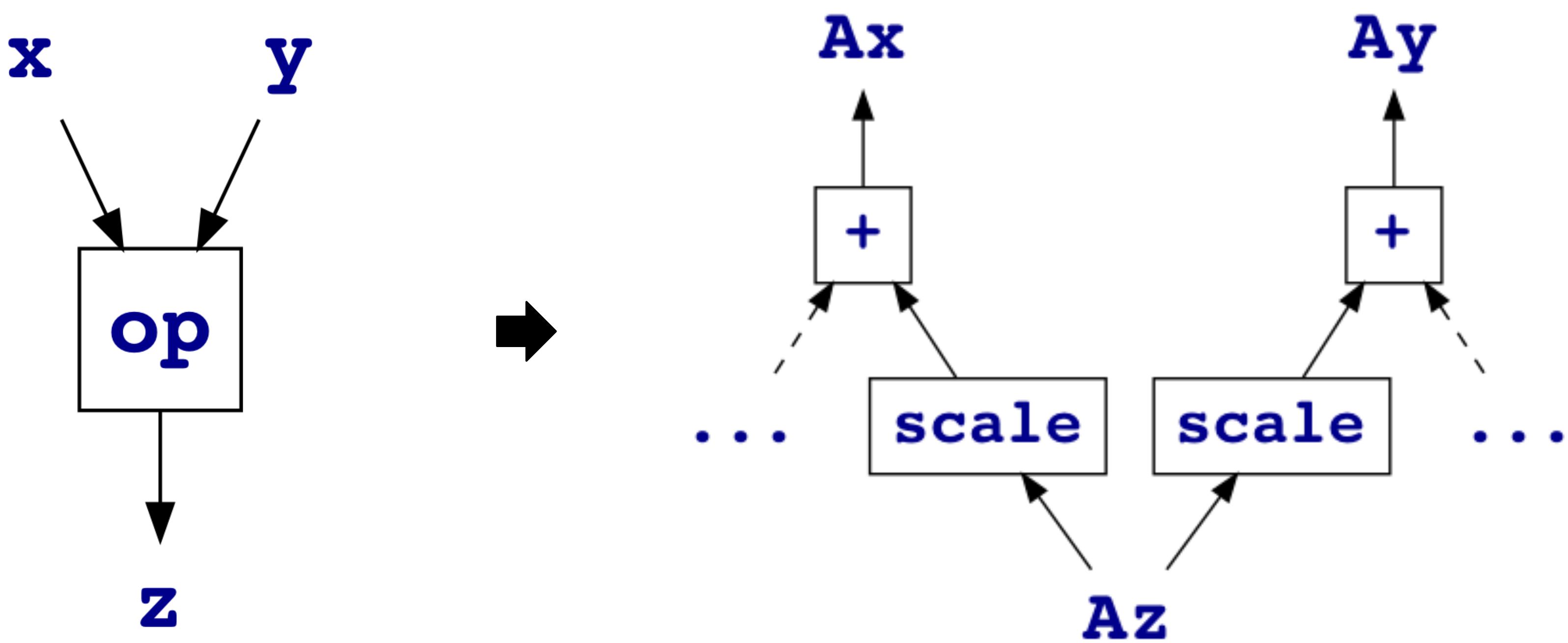
$$\frac{ds}{da} = D_0(*)(a,a)\frac{ds}{dc} + D_1(*)(a, a)\frac{ds}{dc} + D_0(*)(a, b)\frac{ds}{dd}$$

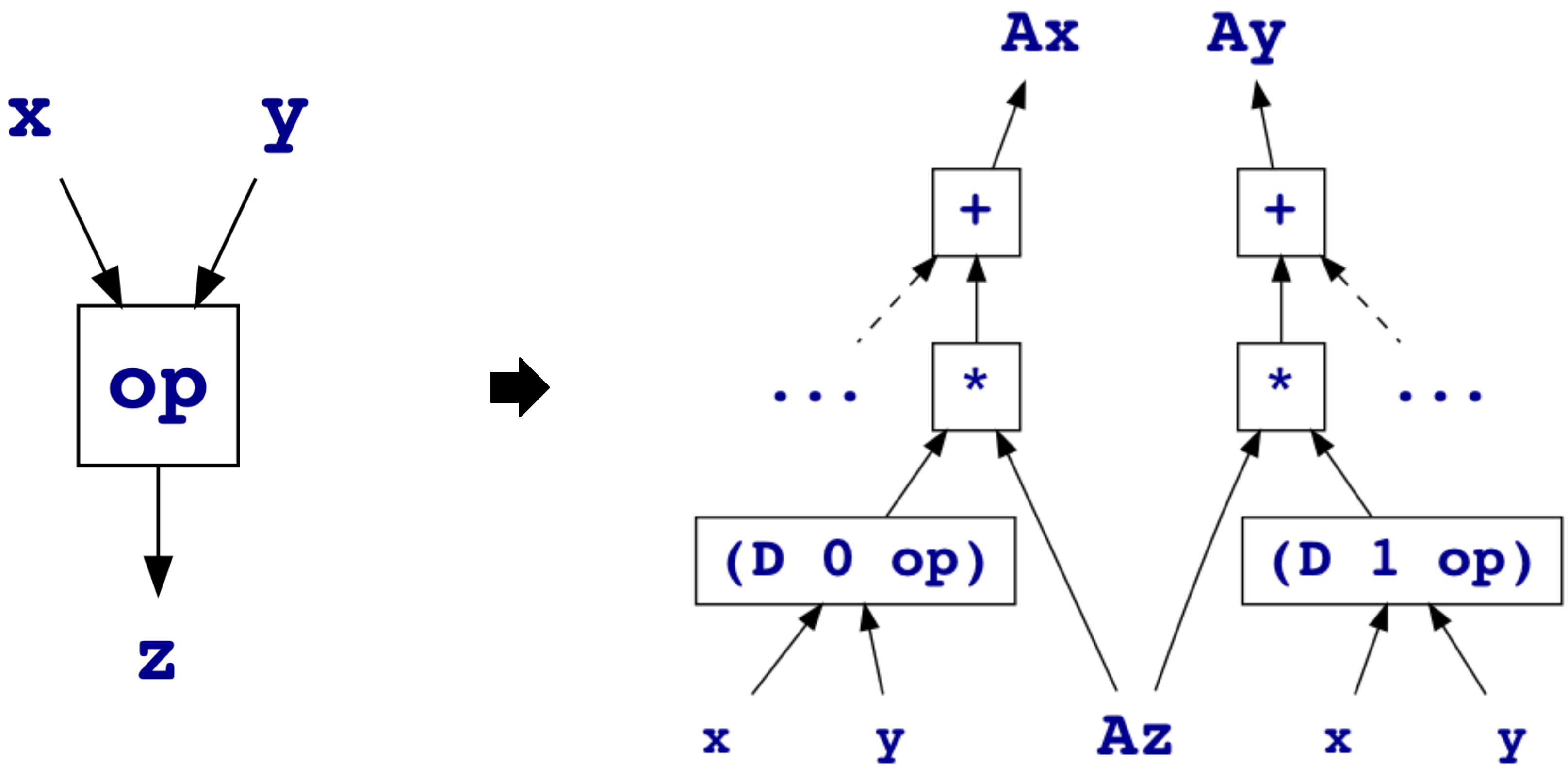
$$Df(a, b) = \left(\frac{ds}{da}, \frac{ds}{db}\right)$$

Reverse mode

Will write `Ax` instead of $\frac{ds}{dx}$

Known as *sensitivity variables* or *adjoints*


x
y
op
z
Ax
Ay
+
+
...
scale
scale
...
Az


x
y
op
z
Ax
Ay
+
+
. . .
*
*
. . .
(D 0 op)
(D 1 op)
x
y
Az
x
y

We can now differentiate any *expression*

involving *primitive operations*

Idea: the return value of a function was determined from a particular (dynamic) call graph.

Differentiate *that*

Options:

- runtime trace
- static code transformation
- local transformations: dual numbers or continutations

Options:

- **runtime trace**
- static code transformation
- local transformations: dual numbers or continutations

Tracing program execution

We want a *flat* trace, which:

contains only *primitive operations*

```
  (sum-squares x y)
=> (sum-squares 3 4)
=> 25
```

```
(sum-squares x y)

=> (sum-squares 3 4)

=> 25
```

```
x
```

=> 3, as

t1	(constant 3)	3

```
y
```

=> 4, as

t2	(constant 4)	4

```
(sum-squares x y)
```

=> 25, as

t1	(constant 3)	3
t2	(constant 4)	4
t3	(app * t1 t1)	9
t4	(app * t2 t2)	16
t5	(app + t3 t4)	25

Let’s make a little language that does this...

assignments

```
(struct assignment (id expr val))
```

assignments

```
(struct assignment (id expr val)
  #:guard (struct-guard/c symbol? expr? any/c))
```

assignments

```
(struct assignment (id expr val)
  #:guard (struct-guard/c symbol? expr? any/c))

     (define (expr? e)
       (match e
         [(list 'constant _) #t]
         [(list 'app (? symbol? _) ..1) #t]
         [_ #f]))
```

trace

```
(struct trace (assignments))
```

trace

```
(struct trace (assignments))

   (trace-add tr assgn)
   (trace-append trs ...)
   (trace-get tr id)
```

trace

```
(struct trace (assignments))

    (trace-add tr assgn)
    (trace-append trs ...)
    (trace-get tr id)
```

top of a trace is the most recent assignment

```
(top tr)
```

trace

```
(struct trace (assignments))

    (trace-add tr assgn)
    (trace-append trs ...)
    (trace-get tr id)
```

top of a trace is the most recent assignment

```
(top tr)

(top-val tr)
(top-id tr)
(top-expr tr)
```

trace-lang functions

```
(define (+& a b)
  (trace-add
   (trace-append a b)
   (make-assignment
    #:expr (list 'app '+ (top-id a) (top-id b))
    #:val  (+ (top-val a) (top-val b)))))
```

trace-lang functions

```
(define (*& a b)
  (trace-add
   (trace-append a b)
   (make-assignment
    #:expr (list 'app '* (top-id a) (top-id b))
    #:val  (* (top-val a) (top-val b)))))
```

trace-lang functions

```
(define (exp& x)
  (trace-add
   x
   (make-assignment
    #:expr (list 'app 'exp (top-id x))
    #:val  (exp (top-val x)))))
```

```
(define (f a ...)
  (trace-add
   (trace-append a ...)
   (make-assignment
    #:expr (list 'app f-name (top-id a) ...)
    #:val  (let ([a (top-val a)] ...)
             body ...))))
```

```
(define-syntax-rule
  (define-traced-primitive (f a ...) f-name
   body ...)
    (define (f a ...)
      (trace-add
       (trace-append a ...)
       (make-assignment
        #:expr (list 'app f-name (top-id a) ...)
        #:val  (let ([a (top-val a)] ...)
                 body ...)))))
```

```
(define-syntax-rule
  (define-traced-primitive (f a ...) f-name
    body ...)
     (define (f a ...)
       (trace-add
        (trace-append a ...)
        (make-assignment
         #:expr (list 'app f-name (top-id a) ...)
         #:val  (let ([a (top-val a)] ...)
                     body ...)))))
```

trace-lang functions

```
(define-traced-primitive (+& a b) '+
  (+ a b))
(define-traced-primitive (*& a b) '*
  (* a b))
; ...
(define-traced-primitive (<& a b) '<
  (< a b))
; ...
(define-traced-primitive (cons& a b) 'cons
  (cons a b))
; ...
```

```
#lang racket
; ...
(provide (rename-out [+& +]
                     [*& *]
                     [exp& exp]
                     ...))
; ...
```

```
(define-syntax-rule (define& (f args ...)
                         body ...)
  (define (f args ...)
    (trace-append (let () body ...)
                  args ...)))
```

```
(define-syntax-rule (if& test-expr
                         then-expr
                         else-expr)
  (if (top-val test-expr)
      then-expr
      else-expr))
```

```
#lang rackpropagator/trace
; ...
```

```
(+ 1 2)
```

```
(+ 1 2)

; trace-items: contract violation
; expected: trace?
; given: 1
```

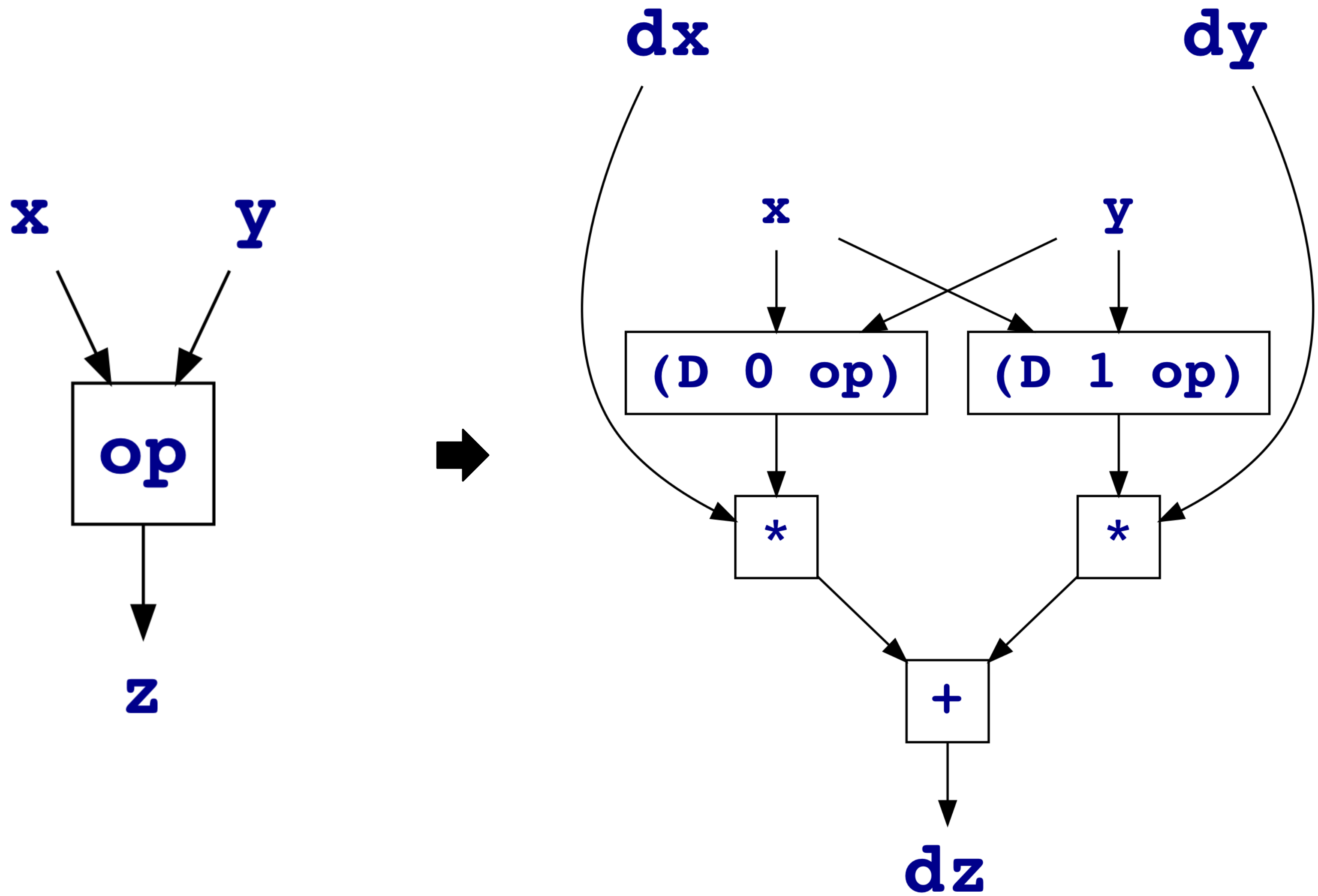
Interposition points

```
(+ 1 2)
=> (#%app + (#%datum . 1) (#%datum . 2))
```

```
(datum& . 1)
=> (make-trace (make-assignment #:val 1))
```

```
(provide (rename-out [datum& #%datum]))
```

Recap: Forward-mode AD



Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x          (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
        (let ([dz (d-prim-op z x indep-ids
                             tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))})))
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
       (let ([dz (d-prim-op z x indep-ids
                                      tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))})))))
```

Forward-mode AD

```
(for/fold ([sum 0]
           [prod 1])
          ([x (range 1 6)])
  (values (+ x sum)
          (* x prod)))
=>
15
120
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
        (let ([dz (d-prim-op z x indep-ids
                                  tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))})))))
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result     (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
        (let ([dz (d-prim-op z x indep-ids
                                 tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))})))
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
        (let ([dz (d-prim-op z x indep-ids
                             tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))}))))
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
        (let ([dz (d-prim-op z x indep-ids
                                 tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))}))))
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
        (let ([dz (d-prim-op z x indep-ids
                             tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))})))))
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
     (let ([dz (d-prim-op z x indep-ids
                              tr deriv-dict)])
        {values
         (trace-append dz tr)
         (hash-set deriv-dict
                   (id z) (top-id dz))})))))
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
        (let ([dz (d-prim-op z x indep-ids
                                    tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))}))))
```

Forward-mode AD

```
(define ((partial/f i f) . xs)
  (let ([x           (top-id (list-ref xs i))]
        [indep-ids (map top-id xs)]
        [result    (apply f xs)])
    (define-values (Dresult _)
      (for/fold ([tr result]
                 [deriv-dict (hash)])
                ([z (reverse (trace-items result))])
        (let ([dz (d-prim-op z x indep-ids
                             tr deriv-dict)])
          {values
           (trace-append dz tr)
           (hash-set deriv-dict
                     (id z) (top-id dz))}))))
```

```
; d-prim-op: assignment? symbol? (Listof symbol?)
;   trace? (HashTable symbol? symbol?) -> trace?
(define (d-prim-op z x-symb indep-ids
                   tr deriv-dict)

  ; d : symbol? -> trace?
  (define (d s)
    (trace-get tr (hash-ref deriv-dict s)))

  (cond
    ; ...
    ))
```

```
; ...
(cond
  [(eq? (id z) x-symb) (datum& . 1.0)]
  [(memq (id z) indep-ids) (datum& . 0.0)]
  [else
   (match (expr z)
     ; ...
     )])
```

```
; ...
(match (expr z)
  [(list 'constant null)  (datum& . null)]
  [(list 'constant c)     (datum& . 0.0)]
  ; ...
  )
```

```
; ...
(match (expr z)
  ; ...
  [(list 'app op xs ...)
   (let ([xs& (map (curry trace-get tr) xs)])
     (for/fold ([acc (datum& . 0.0)])
               ([x xs]
                [i (in-naturals)])
       (define D_i_op (apply (partial i op) xs&))
       (+& (*& D_i_op (d x)) acc)))])
```

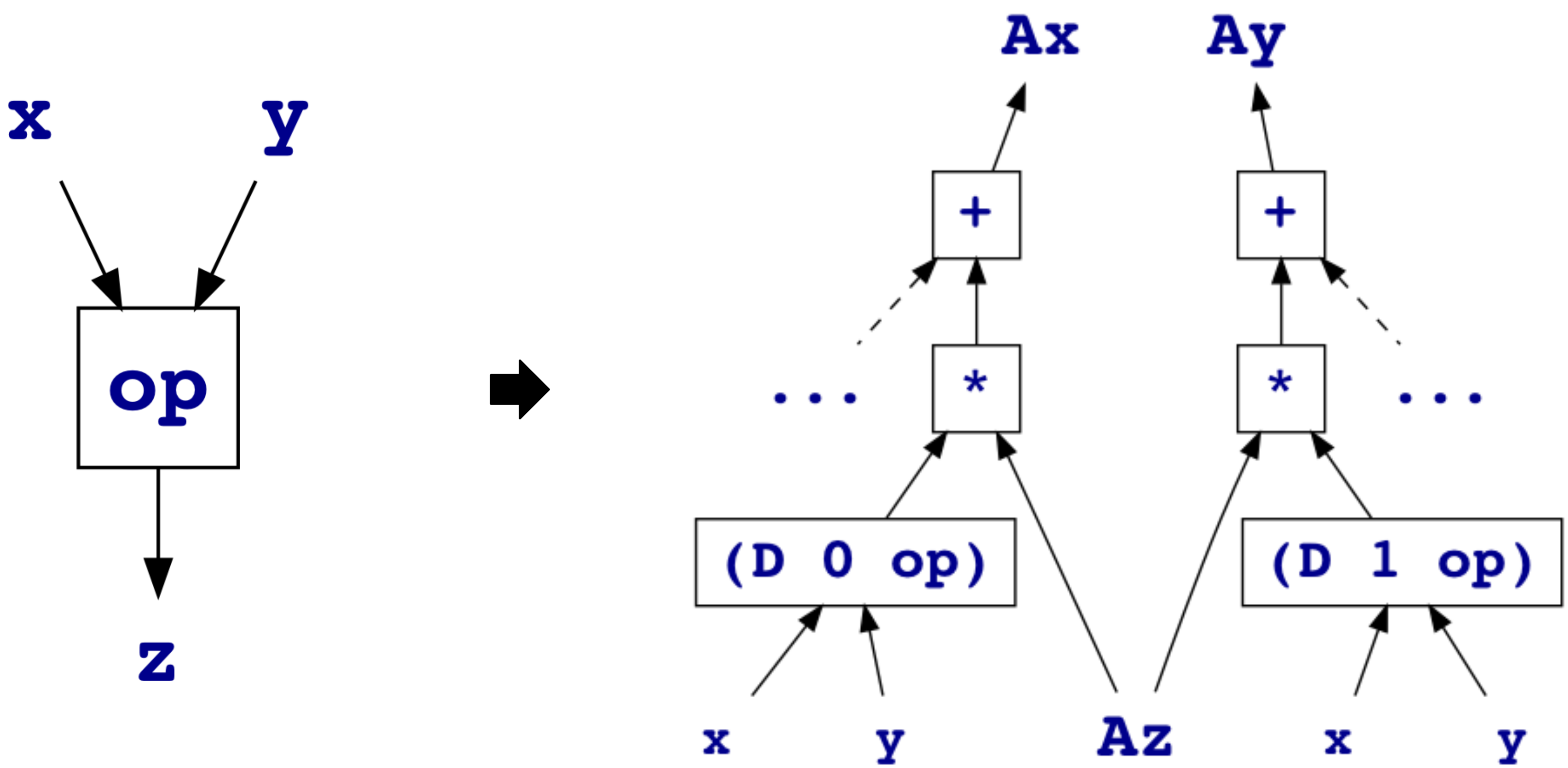
```
((D cons) (f x) (g y))
= (cons ((D f) x) ((D g) y))
```

```
((D car) (cons (f x) (g y)))
= ((D f) x)
```

```
((D cdr) (cons (f x) (g y)))
= ((D g) y)
```

```
; ...
(match (expr z)
  ; ...
  [(list 'app 'cons x y) (cons& (d x) (d y))]
  [(list 'app 'car ls)   (car&  (d ls))]
  [(list 'app 'cdr ls)   (cdr&  (d ls))]
  ; ...
  )
```

Recap: Reverse-mode AD



```
(define (A/r result-tr indep-ids s)
  (define seed-id (top-id result-tr))
  (define seed-tr (trace-append s result-tr))

  (define-values (tr _ adjoints)
    (for/fold ([tr seed-tr]
               [adjoint-terms
                (hash seed-id
                      (list (top-id seed-tr)))]
               [adjoints (hash)])
              ([w (trace-items result-tr)])

      ; ...
      ))
  ; ...
  )
```

```
(define (A/r result-tr indep-ids s)
  (define seed-id (top-id result-tr))
  (define seed-tr (trace-append s result-tr))

  (define-values (tr _ adjoints)
    (for/fold ([tr seed-tr]
               [adjoint-terms
                (hash seed-id
                      (list (top-id seed-tr)))]
               [adjoints (hash)])
              ([w (trace-items result-tr)])

      ; ...
      ))
  ; ...
  )
```

```
(define (A/r result-tr indep-ids s)
  (define seed-id (top-id result-tr))
  (define seed-tr (trace-append s result-tr))

  (define-values (tr _ adjoints)
    (for/fold ([tr seed-tr]
               [adjoint-terms
                (hash seed-id
                      (list (top-id seed-tr)))]
               [adjoints (hash)])
              ([w (trace-items result-tr)])

      ; ...
      ))
  ; ...
  )
```

```
(for/fold (...)
          ([w (trace-items result-tr)])
  (define Aw-terms
    (map (curry trace-get tr)
         (hash-ref adjoint-terms (id w))))
  (define Aw
    (trace-append
     (foldl cons-add (car Aw-terms) (cdr Aw-terms))
     tr))
  (define-values (tr* adjoint-terms*)
    (A-prim-op w Aw adjoint-terms))
  {values tr*
          adjoint-terms*
          (hash-set adjoints (id w) (top-id Aw))})
```

```
(for/fold (...)
          ([w (trace-items result-tr)])
  (define Aw-terms
    (map (curry trace-get tr)
         (hash-ref adjoint-terms (id w))))
  (define Aw
    (trace-append
     (foldl cons-add (car Aw-terms) (cdr Aw-terms))
     tr))
  (define-values (tr* adjoint-terms*)
    (A-prim-op w Aw adjoint-terms))
  {values tr*
          adjoint-terms*
          (hash-set adjoints (id w) (top-id Aw))})
```

```
(for/fold (...)
          ([w (trace-items result-tr)])
  (define Aw-terms
    (map (curry trace-get tr)
         (hash-ref adjoint-terms (id w))))
  (define Aw
    (trace-append
     (foldl cons-add (car Aw-terms) (cdr Aw-terms))
     tr))
  (define-values (tr* adjoint-terms*)
    (A-prim-op w Aw adjoint-terms))
  {values tr*
          adjoint-terms*
          (hash-set adjoints (id w) (top-id Aw))})
```

```
(cons-add '(1 2 (3) . 4)
          '(0 1 (2) . 3))
=> '(1 3 (5) . 7)

(cons-zero '(1 () (2) . 4))
=> '(0 () (0) . 0)
```

```
(for/fold (...)
          ([w (trace-items result-tr)])
  (define Aw-terms
    (map (curry trace-get tr)
         (hash-ref adjoint-terms (id w))))
  (define Aw
    (trace-append
     (foldl cons-add (car Aw-terms) (cdr Aw-terms))
     tr))
  (define-values (tr* adjoint-terms*)
    (A-prim-op w Aw adjoint-terms))
  {values tr*
          adjoint-terms*
          (hash-set adjoints (id w) (top-id Aw))})
```

```
(for/fold (...)
          ([w (trace-items result-tr)])
  (define Aw-terms
    (map (curry trace-get tr)
         (hash-ref adjoint-terms (id w))))
  (define Aw
    (trace-append
     (foldl cons-add (car Aw-terms) (cdr Aw-terms))
     tr))
  (define-values (tr* adjoint-terms*)
    (A-prim-op w Aw adjoint-terms))
  {values tr*
          adjoint-terms*
          (hash-set adjoints (id w) (top-id Aw))})
```

```
; ...
(let* ([tr* (trace-add
             tr
             (make-assignment #:val 0.0))]
       [zero-id (top-id tr*)])
  (trace-prune
   (apply
    list&
    (for/list ([x indep-ids])
      (trace-get
       tr*
       (hash-ref adjoints x zero-id))))))
```

```
w ← (cons x y)
```

=>

```
Ax ← (car Aw)
Ay ← (cdr Aw)
```

w ← (car xs)

=>

(car Axs) ← Aw

$$w \leftarrow (\mathtt{cdr}\ xs)$$

=>

$$(\mathtt{cdr}\ Axs) \leftarrow Aw$$

```
w ← (car xs)
```

=>

```
Axs ← (cons Aw (cons-zero (cdr xs)))
```

```
w ← (cdr xs)
```

=>

```
Axs ← (cons (cons-zero (car xs)) Aw)
```

```
(define (A-prim-op w Aw adjoint-terms)

  (match (expr w)
    ; ...
    [(list 'app 'cons x y)
     (let ([Ax (car& Aw)]
           [Ay (cdr& Aw)])
       {values (trace-append Ay Ax Aw)
               (upd-adj adjoint-terms
                        x Ax
                        y Ay)})]
    ; ...
    ))
```

```
(define (A-prim-op w Aw adjoint-terms)

  (match (expr w)
    ; ...
    [(list 'app 'car xs)
     (let ([xs& (trace-get Aw xs)]
           [tr (cons& Aw (cons-zero (cdr& xs&)))])
       {values (trace-append tr Aw)
               (upd-adj adjoint-terms xs tr)})]
    ; ...
    ))
```

http://github.com/ots22/rackpropagator

References

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From automatic differentiation to message passing

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PAPER

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Conal Elliot

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PAPER

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Fei Wang *et al.* (2018)

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`https://beautifulracket.com/`

Matthew Butterick

References

BOOK

Structure and Interpretation of Classical Mechanics (2nd ed.)

`https://mitpress.mit.edu/sites/default/files↙`

`/titles/content/sicm_edition_2/book.html`

Gerald Jay Sussman & Jack Wisdom (2015)

Program transformation

Can apply the previous work to straight-line code, at compile time

`define` instead of **`assignment`**

Program transformation

```
#lang rackpropagator/↙
  straightline
(define (f x y)
  (define a (+ x y))
  (define b (+ a a))
  (define c (* a y))
  (define d 1.0)
  (+ c d))
```

➡

```
(define (Df x y)
  (define a (+ x y))
  (define t2 1.0)
  (define t3 1.0)
  (define t4 (* t2 t3))
  (define t7 (* t4 y))
  (define t8 (* t4 a))
  (define t9 1.0)
  (define t10 (* t7 t9))
  (define t11 1.0)
  (define t12 (* t7 t11))
  (define t17 (+ t8 t12))
  (define t19 '())
  (define t20 (cons t17 t19))
  (cons t10 t20))
```

Program transformation

```
(define-syntax (define/d stx)
  (syntax-case stx ()
    [(_ (f args ...) body ...)
     (with-syntax
       ([(body* ...)
         (handle-assignments #'(args ...)
                             #'(body ...))])
       #'(define (f args ...)
           body* ...))]))
```

Program transformation

```
(define-syntax (define/d stx)
  (syntax-case stx ()
    [(_ (f args ...) body ...)
     (with-syntax
       ([(body* ...)
         (handle-assignments #'(args ...)
                             #'(body ...))])
       #'(define (f args ...)
           body* ...))]))

  (provide (rename-out [define/d define]))
```

Dual numbers

Dual numbers

sum-of-squares:

Given **a** and **b**

```
c ← (* a a)
d ← (* b b)
e ← (+ c d)
```

Dual numbers

sum-of-squares:

Given `a` and `b`

```
c ← (* a a)
d ← (* b b)
e ← (+ c d)
```

The "forward-mode" transformation:

```
dc ← (+ (* a da) (* da a))
dd ← (+ (* b db) (* db b))
de ← (+ dc dd)
```

Dual numbers

sum-of-squares:

Can interleave the operations computing *x* and **`d`***x*

```
c  ← (* a a)
dc ← (+ (* a da) (* da a))
d  ← (* b b)
dd ← (+ (* b db) (* db b))
e  ← (+ c d)
de ← (+ dc dd)
```

- **`dx`** depends on **`dy`** if and only if **`x`** depends on **`y`**
- **`dx`** depends on **`y`** only if **`x`** depends on **`y`**

Dual numbers

Idea: treat the pair of `x` and `dx` as a single entity. Define combined operations.

Dual numbers

```
(struct dual-number (p d))
```

Dual numbers

```
(struct dual-number (p d))

(define (primal x)
  (cond
    [(dual-number? x) (dual-number-p x)]
    [(number? x) x]
    [else (raise-argument-error
           'primal "number? or dual-number?" x)]))
```

Dual numbers

```
(struct dual-number (p d))

(define (dual x)
  (cond
    [(dual-number? x) (dual-number-d x)]
    [(number? x) (zero x)]
    [else (raise-argument-error
            'dual "number? or dual-number?" x)]))
```

Dual numbers

```
(define (dual-+ x y)
  (if (or (dual-number? x) (dual-number? y))
      (dual-number (+ (primal x) (primal y))
                   (+ (dual x) (dual y)))
      (+ x y)))
```

Dual numbers

```
(define (dual-+ x y)
  (if (or (dual-number? x) (dual-number? y))
      (dual-number (+ (primal x) (primal y))
                   (+ (dual x) (dual y)))
      (+ x y)))
```

Dual numbers

```
(define (dual-* x y)
  (if (or (dual-number? x) (dual-number? y))
      (dual-number (* (primal x) (primal y))
                   (+ (* (dual x) (primal y))
                      (* (primal x) (dual y))))
      (* x y)))
```

Dual numbers

```
(define (dual-* x y)
  (if (or (dual-number? x) (dual-number? y))
      (dual-number (* (primal x) (primal y))
                   (+ (* (dual x) (primal y))
                      (* (primal x) (dual y))))
      (* x y)))
```

Dual numbers

```
; ...
(define (dual-log x)
  (if (dual-number? x)
      (dual-number (log (primal x))
                   (/ (dual x) (primal x)))
      (log x)))
; ...
```

Dual numbers

- **only** need to define the primitive numerical functions
- Can be implemented with operator overloading
- A **local** program transformation

Dual numbers: Differentiation

```
(define ((D n f) . args)
  (let ([args* (for/list [(i (in-naturals))
                          (a args)]
                 (if (= i n)
                     (dual-number a 1)
                     (dual-number a 0)))])
    (get-dual-part (apply f args*))))
```

Dual numbers: Differentiation

```
(define ((D n f) . args)
  (let ([args* (for/list [(i (in-naturals))
                          (a args)]
                 (if (= i n)
                     (dual-number a 1)
                     (dual-number a 0)))])
    (get-dual-part (apply f args*))))
```

Dual numbers: Differentiation

```
(define ((D n f) . args)
  (let ([args* (for/list [(i (in-naturals))
                          (a args)]
                 (if (= i n)
                     (dual-number a 1)
                     (dual-number a 0)))])
    (get-dual-part (apply f args*))))
```

Dual numbers: Differentiation

```
(define ((D n f) . args)
  (let ([args* (for/list [(i (in-naturals))
                          (a args)]
                 (if (= i n)
                     (dual-number a 1)
                     (dual-number a 0)))])
    (get-dual-part (apply f args*))))
```

Dual numbers: Differentiation

```
(define ((D n f) . args)
  (let ([args* (for/list [(i (in-naturals))
                          (a args)]
                 (if (= i n)
                     (dual-number a 1)
                     (dual-number a 0)))])
    (get-dual-part (apply f args*))))
```

Dual numbers: Differentiation

```
(define ((D n f) . args)
  (let ([args* (for/list [(i (in-naturals))
                          (a args)]
                 (if (= i n)
                     (dual-number a 1)
                     (dual-number a 0)))])
    (get-dual-part (apply f args*))))
```

Helper function:

```
(get-dual-part
 (list (dual-number 0.0 1.0)
       2.0
       (cons (dual-number 3.0 0.0)
             (dual-number 4.0 5.0))))
=> (1.0 0.0 (0.0 . 5.0))
```